Contagiousness in Information Systems

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August 21, 2009

Abstract

Players' higher order beliefs can have a strong impact on possible equilibria in games of incomplete information. For example, higher order uncertainty can leave a game with only one equilibrium. This paper examines how this type of "infection," in which higher order uncertainty makes most of the type space play a specific action, is related to properties of the information system. We show that whether infection occurs depends on if there is a set of types that sufficiently cohesive and sufficiently large that can stop the infection. We find that a type of uniformity across an information system makes it more susceptible to contagion. We also find bounds on how large of an impact a small probability event can have on an information system.

1 Introduction

Previous work has shown that in games of incomplete information, the presence of higher order uncertainty can narrow down the set of possible equilibria. In Rubinstein's Electronic Mail Game [13], communication can allow all players to know that all players know that all players ... know something payoff-relevant for many iterations. Players do not have *common knowledge*, though, and eventually, some player knows that some player knows ... that some player does not know something payoff-relevant. This lack of common knowledge, Rubinstein shows, can rule out certain equilibria that would have been possible with common knowledge. Carlsson and van Damme [1] show that in 2×2 complete information games with two strict equilibria, uncertainty about payoffs can, through players' higher order beliefs, eliminate the risk dominant equilibrium.

This paper examines how players' prior beliefs are related to this notion of "infection," in which uncertainty reduces the number of equilibria. Similar to Rubinstein [13] and Carlsson and van Damme [1], we rely on it being known that there are certain types that have dominant actions. These initially infected types can dictate a unique best response for almost all types. For example, some types (set S_1) may believe that they will play against an initially infected type, and their beliefs may make a single action a best response. Other types (S_2) may believe that they will interact with this set of types (S_1). A further set of types (S_3) may believe they will interact with this set (S_2). Eventually, most types may believe they will play against a type that believes it will play against a type that believes ... that it will play against a type in the infected set. It is also possible that there is only one equilibrium, in which all players play the action of the initially infected set of types, regardless of the signal a player receives.

We use tools from the higher order beliefs and local interactions literatures in order to examine our notion of infection. Belief operators, introduced by Monderer and Samet [6], allow us to examine when a player believes that its opponent believes that its opponent ... believes that something will happen. Morris et al. [10] introduce the *belief potential* of an information system as a measure of its susceptibility to infection. We use a weaker notion of (η, p)-*contagiousness*: rather than measuring the susceptibility of an information system to infection by requiring that *any* set of types can infect all types, we require only that there is *some* set of types that occurs with sufficiently low prior probability (up to η) that can infect a set that occurs with sufficiently high prior probability (at least $1 - \eta$). Morris [9] studied how contagion can occur in dynamic *local interaction games*. He showed that low neighbor growth and a type of uniformity across players' interactions make a local interaction game more susceptible to contagion. We extend some of his tools and results to a setting of incomplete information. We find that an assumption akin to low neighbor growth is not needed, since a similar property is automatically satisfied for games of incomplete information.

After dealing with preliminaries in Section 2, we define two belief operators in Section 3. One operator is monotonically increasing, while the other is monotonically decreasing. We show a duality between both operators, which allows for related interpretations of when a set of types can or cannot infect a large set of types. As we show in Section 4, a set of types can infect a large set of types if the monotonically increasing operator expands by enough. That is, the set of types that believe that it will interact with a type that believes that it will interact with a type that believes ...that it will interact with a type in the initial set occurs with sufficiently high prior probability. Conversely, a set of types cannot infect a large set of types if the complement of the initially infected set contains a set that occurs with sufficiently high prior probability and is sufficiently cohesive. In Section 5.1, we show that a type of uniformity across players' beliefs will make the information system more susceptible to contagion. In Section 5.2, we adapt a result from Oyama and Tercieux [12] to establish an upper bound on the prior probability an infected set can be as a function of the prior probability of the initially infected set. This result provides limits on how large of an impact an unlikely event can have.

2 Preliminaries

We are interested in *information systems* that meet these criteria

- For notational convenience, we restrict our analysis to a game with two players, x and y.
- Associated with each player is a countably infinite set of types X_x and X_y.
- We denote $\mathcal{T} := X_x \cup X_y$ the set of all possible types.
- Possible states or interactions are given by $\mathcal{I} := X_x \times X_y$.
- Information partitions H_x and H_y are induced by each player only knowing her own type.
- Players share a common prior μ over the set of possible interactions $\mathcal{I}$, and each type occurs with strictly positive probability.

An information system is denoted as a tuple $(\mathcal{I}, \{x, y\}, \{H_x, H_y\}, \mu)$ specifying states, players, information partitions, and beliefs. The class of information systems that meet these criteria is denoted $\mathcal{P}$.

We are also interested in *incomplete information games* that have information systems specified as above as well as the following properties

- Each player i has a finite set of actions, A_i.
- Payoffs for player i are given by the action profile and the player's type, with the function $u_i : A_x \times A_y \times X_i \to \mathbb{R}$.

An incomplete information game is a tuple $((\mathcal{I}, \{x, y\}, \{H_x, H_y\}, \mu), \{A_x, A_y\}, \{u_x, u_y\})$ specifying the information system, actions, and payoff functions. Let $\mathcal{G}$ be the class of incomplete information games that meet the criteria above.

A *pure strategy* for player i is a function $s_i : X_i \to A_i$. The set of pure strategies for player i is given by S_i. A *pure strategy profile* $s = (s_x, s_y) \in S_x \times S_y$ specifies a strategy for each player. A *mixed strategy* σ_i for player i is a probability distribution over pure strategies, and a *mixed strategy profile* $\sigma = (\sigma_x, \sigma_y) \in \Sigma_x \times \Sigma_y$ specifies a distribution for each player. We let $t_i(\omega)$ denote the type of player i in state $\omega \in \mathcal{I}$.

Definition 1 (Bayesian-Nash Equilibrium)**.** A strategy profile (σ_x, σ_y) is a *Bayesian-Nash equilibrium*, or simply an *equilibrium*, if for every $\tau_x \in \Sigma_x$,

$$\sum_{\omega \in \mathcal{I}} \mu(\omega) u_x \left(\sigma_x(t_x(\omega)), \sigma_y(t_y(\omega)); t_x(\omega)\right) \geq \sum_{\omega \in \mathcal{I}} \mu(\omega) u_x \left(\tau_x(t_x(\omega)), \sigma_y(t_x(\omega)); t_x(\omega)\right),$$

and analogously for player y.

Since our probability measure μ is specified over states, and we wish to make statements about conditional probabilities that types have of interacting, we introduce a bit more notation:

$$\mathcal{I}(T) := \{\omega \in \mathcal{I} \mid (t, \cdot) = \omega \text{ or } (\cdot, t) = \omega \text{ for some } t \in T\}.$$

That is, $\mathcal{I}(T)$ gives the states in which a type in T plays.

It follows directly from the definition that $\mathcal{I}\left(\bigcup_{k\in\mathbb{N}} T_k\right) = \bigcup_{k\in\mathbb{N}} \mathcal{I}(T_k)$. Further, if $T_{k+1} \subseteq T_k$ for all $k \in \mathbb{N}$, then $\mathcal{I}\left(\bigcap_{k\in\mathbb{N}} T_k\right) = \bigcap_{k\in\mathbb{N}} \mathcal{I}(T_k)$.

We define the prior probability that a type in some set $T \in \mathcal{T}$ will play by

$$\pi(T) := \mu(\mathcal{I}(T)),$$

so that the conditional probability that a type in one set will interact with a type in another set is given by,

$$\pi(T \mid S) := \mu(\mathcal{I}(T) \mid \mathcal{I}(S)).$$

With minor abuse of notation, we will denote for a singleton set $\{t\} \subseteq \mathcal{T}$, $\pi(T \mid t)$ instead of $\pi(T \mid \{t\})$, and in other situations in which no confusion will result. Note that in general, $\pi(T \mid S) + \pi(\overline{T} \mid S) \neq 1$, and that $\pi(T \cap S) = \mu(\mathcal{I}(T \cap S)) \neq \mu(\mathcal{I}(T) \cap \mathcal{I}(S))$.

3 Belief Operators

Our notion of the contagiousness of an information system uses *belief operators*, introduced by Monderer and Samet [6]. The *belief contraction operator*, or simply the *contraction operator*, BC^p, associates with a set of types S the subset of types that believe with probability at least p that they will play against with another type in S. Formally, for $S \subseteq \mathcal{T}$,

$$\begin{aligned} BC^p(S) :=& \{t \in S \mid \pi(S \backslash t \mid t) \geq p\} \\ =& \{t \in \mathcal{T} \mid \pi(S \backslash t \mid t) \geq p \text{ and } t \in S\}. \end{aligned}$$

We introduce the second definition to more easily see the relationship between the belief expansion and its counterpart, which we will introduce shortly.

Lemma 1 (Monotonicity of BC^p)**.** *For sets $T, S \subseteq \mathcal{T}$, if $T \subseteq S$, then $BC^p(T) \subseteq BC^p(S)$.*

Proof. If $t \in BC^p(T)$, then $\pi(T \backslash t \mid t) > p$. Since

$$\mathcal{I}(T \backslash t) \subseteq \mathcal{I}((T \cup S) \backslash t) = \mathcal{I}(S \backslash t),$$

we have $\pi(S \backslash t \mid t) \geq \pi(T \backslash t \mid t) > p$. Thus, $t \in BC^p(S)$. □

Lemma 2 (Continuity of BC^p)**.** *Let $T, T_k \subseteq \mathcal{T}$ for $k \in \mathbb{N}$. If $T_k \downarrow T$, then $BC^p(T) = \bigcap_{k \in \mathbb{N}} BC^p(T_k)$.*

Proof. Since $T \subseteq T_k$ for every k, then by monotonicity, $BC^p(T) \subseteq BC^p(T_k)$ for every k. Thus, $BC^p(T) \subseteq \bigcap_{k \in \mathbb{N}} BC^p(T_k)$.

It remains to show that $\bigcap_{k \in \mathbb{N}} BC^p(T_k) \subseteq BC^p(T)$. Since $(T_k)_{k \in \mathbb{N}} \downarrow T$, $(\mathcal{I}(T_k))_{k \in \mathbb{N}}$ is a decreasing sequence and $\mathcal{I}\left(\bigcap_{k \in \mathbb{N}} T_k\right) = \mathcal{I}(T)$. These latter two facts also imply

$$\mathcal{I}\left(\bigcap_{k \in \mathbb{N}} T_k\right) = \bigcap_{k \in \mathbb{N}} \mathcal{I}(T_k),$$

so we have $\mathcal{I}(T_k) \downarrow \mathcal{I}(T)$. This gives (see, for example, [2], Lemma 1.3.5)

$$\lim_{k \to \infty} \pi(T_k \backslash t \mid t) = \pi(T \backslash t \mid t).$$

If $t \in \bigcap_{k \in \mathbb{N}} BC^p(T_k)$, then $\mu(T_k \backslash t \mid t) \geq p$ for every k. We must then have that $\mu(T \backslash t \mid t) \geq p$, so that $t \in BC^p(T)$. Thus, $\bigcap_{k \in \mathbb{N}} BC^p(T_k) \subseteq BC^p(T)$. □

Note that subpotency (that is, $BC^p(BC^p(T)) \subseteq BC^p(T)$) follows from monotonicity here, and so BC^p is a belief operator in the sense of Monderer and Samet [6].

The contraction operator can be interpreted in terms of higher order beliefs. For example, $BC^p(BC^p(T))$ represents the set of types in T that believe with probability at least p that they will play against a type in T that believes with probability at least p that it will play against a type in T. This process can be iterated any number of times to obtain the set of types in T that believe with probability at least p that they will play against a type that believes with probability at least p that ... believes with probability at least p that it is in T. The following operator allows for any number of these iterations:

$$CC^p(T) := \bigcap_{k \in \mathbb{N}} (BC^p)^k(T).$$

By definition, $BC^p(T) \subseteq T$. If also $T \subseteq BC^p(T)$, then we say T is *p-closed under contraction*.

Proposition 1. *For any set $T \subseteq \mathcal{T}$, $CC^p(T)$ is p-closed under contraction.*

Proof. $\left((BC^p)^k(T)\right)_{k\in\mathbb{N}} \downarrow CC^p(T)$, so that by continuity,

$$BC^p(CC^p(T)) = \bigcap_{k\in\mathbb{N}} BC^p\left((BC^p)^k(T)\right) = \bigcap_{k\geq 2}(BC^p)^k(T) \supseteq \bigcap_{k\in\mathbb{N}}(BC^p)^k(T) = CC^p(T).$$

□

The CC^p operator ("closed under contraction") allows us to make statements both about higher order beliefs and about the cohesiveness of a set of types. Types in $CC^p(T)$ will both interact with another type in $CC^p(T)$ with probability at least p, and believe with probability at least p that they will interact with a type that believes with probability at least p that it will interact with a type that believes with probability at least p ... that it will interact with a type in $CC^p(T)$ for any number of iterations (cf. Kets [5] on this). It can be shown that for a set $T \subseteq \mathcal{T}$, $CC^p(T)$ is the largest subset of T that is p-closed under contraction (see Proposition 1 in Morris [8]).

Related to the contraction operator is the *belief expansion operator*, or simply the *expansion operator*, BE^p, which associates with a set of types T the set of types S that believe with conditional probability greater than p that they will interact with a type in T. Formally, for $S \in \mathcal{T}$, and $p \in [0, 1)$,

$$BE^p(S) := \{t \in \mathcal{T} \mid \pi(S \mid t) > p\}.$$

Whenever $t \notin T$, we have $\pi(T\backslash t \mid t) = \pi(T \mid t)$, and if $t \in T$, then $\pi(T \mid t) = 1$. Thus, we have the equivalent definition,

$$BE^p(S) := \{t \in \mathcal{T} \mid \pi(S\backslash t \mid t) > p \text{ or } t \in S\}.$$

We will use each of these definitions for the proofs that follow. Note that $T \subseteq BE^p(T)$. We now prove results analogous to those of the contraction operator.

Lemma 3 (Monotonicity of BE^p)**.** *For sets T, $S \subseteq \mathcal{T}$, if $T \subseteq S$, then $BE^p(T) \subseteq BE^p(S)$.*

Proof. If $t \in BE^p(T)$, then $\pi(T \mid t) > p$. Since

$$\mathcal{I}(T) \subseteq \mathcal{I}(T \cup S),$$

we see $\pi(T \cup S \mid t) > p$. And since $T \cup S = S$, we have $\pi(S \mid t) > p$. Thus, $t \in BE^p(S)$. □

Lemma 4 (Continuity of BE^p)**.** *Let $T, T_k \subseteq \mathcal{T}$ for $k \in \mathbb{N}$. If $T_k \uparrow T$, then $BE^p(T) = \bigcup_{k\geq 1} BE^p(T_k)$.*

Proof. Since $T_k \subseteq T$ for every k, then by monotonicity, $BE^p(T_k) \subseteq BE^p(T)$ for every k. Thus, $\bigcup_{k\in\mathbb{N}} BE^p(T_k) \subseteq BE^p(T)$.

It remains to show that $BE^p(T) \subseteq \bigcup_{k\in\mathbb{N}} BE^p(T_k)$. If $t \in BE^p(T)$, then $\pi(T \mid t) > p$. There is then some $r > p$ such that $\pi(T \mid t) = r$. If $(T_k)_{k\in\mathbb{N}} \uparrow T$, then $(\mathcal{I}(T_k))_{k\in\mathbb{N}}$ is an increasing sequence and $\mathcal{I}\left(\bigcup_{k\in\mathbb{N}} T_k\right) = \mathcal{I}(T)$. Since, $\mathcal{I}\left(\bigcup_{k\in\mathbb{N}} T_k\right) = \bigcup_{k\in\mathbb{N}} \mathcal{I}(T_k)$, we have $\mathcal{I}(T_k)_{k\in\mathbb{N}} \uparrow \mathcal{I}(T)$. It then follows that

$$\lim_{k\to\infty} \mu(\mathcal{I}(T_k)) = \mu(\mathcal{I}(T)),$$

and so

$$\lim_{k\to\infty} \pi(T_k \mid t) = \pi(T \mid t).$$

We must have some j such that $\pi(T \mid t) - \pi(T_j \mid t) < \frac{r-p}{2}$. It follows that $\pi(T_j \mid t) > \frac{r+p}{2} > p$, so that $t \in BE^p(T_j)$. Thus, $t \in \bigcup_{k\in\mathbb{N}} BE^p(T_k)$, and so $BE^p(T) \subseteq \bigcup_{k\in\mathbb{N}} BE^p(T_k)$. □

Note that BE^p is not a belief operator as defined by Monderer and Samet [6]. Instead of being subpotent and continuous for decreasing sequences, it is superpotent and continuous for increasing sequences.

Iterations of the expansion operator also can also be interpreted in terms of higher order beliefs. For example, $BE^p(BE^p(T))$ represents the set of types that believe with probability greater than p that they will interact with a type that believes with probability greater than p that it will interact with a type in T. This process can be iterated any number of times, which leads us to the following operator:

$$CE^p(T) := \bigcup_{k\in\mathbb{N}} (BE^p)^k(T).$$

By construction, $T \subseteq BE^p(T)$. In the case that $BE^p(T) \subseteq T$, we say that T is *p-closed under expansion.*

Proposition 2. *For any set $T \subseteq \mathcal{T}$, $CE^p(T)$ is p-closed under expansion.*

Proof. $\left((BE^p)^k(T)\right)_{k\in\mathbb{N}} \uparrow CE^p(T)$, so that by continuity,

$$BE^p(CE^p(T)) = \bigcup_{k\in\mathbb{N}} BE^p\left((BE^p)^k(T)\right) = \bigcup_{k\geq 2}(BE^p)^k(T) \subseteq \bigcup_{k\in\mathbb{N}}(BE^p)^k(T) = CE^p(T).$$

□

A useful relationship between the expansion and contraction operators is the following result, which follows directly from the second definitions given for each operator (see also Monderer and Samet [7]).

Proposition 3. *Let $T \subseteq \mathcal{T}$. Then $(BE^p)^k(T) = \overline{(BC^{1-p})^k(\overline{T})}$ for every $k \in \mathbb{N}$. Moreover, $CE^p(T) = \overline{CC^{1-p}(\overline{T})}$.*

Proof. Let $t \in BE^p(T)$. Then either $\pi(T\backslash t \mid t) > p$ or $t \in T$. Thus, either $\pi(\overline{T}\backslash t \mid t) < 1-p$ or $t \notin \overline{T}$. It follows directly from the second definition of BC that $t \notin BC^{1-p}(\overline{T})$. Now let $t \notin BC^{1-p}(\overline{T})$. Then $\pi(\overline{T}\backslash t \mid t) \geq 1-p$ and $t \in \overline{T}$. This implies $\pi(T\backslash t \mid t) \leq p$ and $t \notin T$, so that $t \notin BE^p(T)$. We can thus conclude $BE^p(T) = \overline{BC^{1-p}(\overline{T})}$.

Now suppose that for some $k \in \mathbb{N}$, $(BE^p)^k(T) = \overline{(BC^{1-p})^k(\overline{T})}$. We wish to show that $(BE^p)^{k+1}(T) = \overline{(BC^{1-p})^{k+1}(\overline{T})}$. The first part of the proof allowed for an arbitrary set T, so it follows directly that $BE^p\left((BE^p)^k(T)\right) = \overline{BC^{1-p}\left((BC^{1-p})^k(\overline{T})\right)}$, so we get $(BE^p)^{k+1}(T) = \overline{(BC^{1-p})^{k+1}(\overline{T})}$.

Since $(BE^p)^k(T) = \overline{(BC^{1-p})^k(\overline{T})}$ for every $k \in \mathbb{N}$,

$$\bigcup_{k\in\mathbb{N}}(BE^p)^k(T) = \bigcup_{k\in\mathbb{N}}\overline{(BC^{1-p})^k(\overline{T})},$$

which implies by De Morgan's law that $CE^p(T) = \overline{CC^{1-p}(\overline{T})}$.

□

Corollary 1. *For every $T \subseteq \mathcal{T}$, $\pi(CE^p(T)) = 1 - \pi(CC^{1-p}(\overline{T}))$.*

Recall that the set $CC^{1-p}(\overline{T})$ is the largest subset of $\overline{T}$ that is $(1-p)$-closed under contraction. Proposition 3 shows that a set of types will expand under iterations of BE^p to encompass all types that are not in the set $CC^{1-p}(\overline{T})$. Thus, for an arbitrary set S, $CE^p(S)$ can be interpreted as usual in terms of higher order beliefs as well as in terms of cohesion: $CE^p(S)$ is cohesive in the sense that no type in its complement believes it will interact with a type in $CE^p(S)$ with probability greater than p.

4 Contagiousness

Imagine now that for some game $G \in \mathcal{G}$, there is a player i with a nonempty set of types $I_i \subseteq X_i$ such that for types in I_i, the action L_i is strictly dominant. We then say that I_i is a *prior infected set for player i*. We call the union of the infected sets for both players simply the *prior infected set*, I. We are interested in properties of information systems under which knowledge that types in I play an action L_x or L_y (depending on which player corresponds to the type) will, through iterative strict dominance, force a set of types that occurs with large prior probability to play action L_x or L_y due to their higher order beliefs. Our concept of *posterior* infection narrows down possible actions for different types, through players' higher order beliefs. We formalize our notion of the contagiousness of an information system with the following definition, which is related to the *belief potential* of an information system introduced in Morris, et al. [10].

Definition 2 ((η,p)-Contagiousness)**.** An information system $P \in \mathcal{P}$ is (η,p)*-contagious* if there exists a set $I \subseteq \mathcal{T}$ such that $\pi(I) \leq \eta$ and $\pi(CE^p(I)) \geq 1-\eta$. If P is (η,p)*-contagious* for every $\eta > 0$, then we say it is p-contagious.

Unlike the belief potential, which requires infection of the entire set of types starting from any type, (η,p)-contagiousness is a considerably weaker property, since we only require infection of a sufficiently large set of types occurs starting from a sufficiently small set of types.

We use the notion of *p-dominance* introduced by Morris et al. [10]. This allows us to conclude when an action is a best response to another action, given a player's beliefs.

Definition 3 (p-Dominance)**.** For a game $G \in \mathcal{G}$, an action pair $(a_x, a_y) \in A_x \times A_y$ is *strongly p-dominant in $T \subseteq \mathcal{T}$* if for any $b_x \in A_x$,

$$\sum_{b_y \in A_y} \lambda_y(b_y) g_x(a_x, b_y; t_x) > \sum_{b_y \in A_y} \lambda_y(b_y) g_x(b_x, b_y; t_x),$$

for every type $t_x \in T$ and probability distribution λ_y on A_y such that $\lambda_y(a_y) \geq p$, and analogously for player y.

In the case that the above statement does not hold, but does when $\lambda_j(a_j) > p$, we say that (a_x, a_y) is *weakly p-dominant in T*, for $T \subseteq \mathcal{T}$. We say that an action pair is *p-dominant in T* if it is either strongly or weakly p-dominant in T.

It follows from the definition that if an action pair is p-dominant, then for any q such that $1 \geq q \geq p$, the action pair is q-dominant as well. The definition further precludes two actions pairs that contain one of the same action from being dominant. For example, if (a_x, a_y) and (b_x, a_y) are p and q-dominant, respectively, for $a_x \neq b_x$, then a_x and b_x are both a strict best response to a_y whenever probability greater than $\max\{p, q\}$ is assigned to player y playing a_y. This is inherently contradictory, since both cannot be strict best responses. Strong p-dominance has a natural interpretation when p is 0 or 1. If an action pair (a_x, a_y) is strongly 0-dominant, then a_x and a_y are both strictly dominant actions for players x and y, respectively: no matter what probability distribution player i assumes over actions by player j, action a_i is a strict best response. And if an action pair (a_x, a_y) is 1-dominant, then it is a strict Nash equilibrium.

We are interested primarily in the susceptibility of an information structure to infection. Proposition 4, though, relates contagiousness of an information structure with infection in a particular game of incomplete information.

For convenience in the results that follow, we will say for example that a type t or set of types plays an action L_i when we mean that $\sigma_x(t) = L_x$ or $\sigma_y(t) = L_y$, depending on whether $t \in X_x$ or $t \in X_y$, and similarly when we say an opponent's type plays action L_j. We use the fact that X_x and X_y are disjoint to avoid statements of the form "In any equilibrium σ, for either player x or y, if $t \in CE^p(I) \cap X_x$, then $\sigma_x(t) = L_x$, and if $t \in CE^p(I) \cap X_y$, then $\sigma_y(t) = L_y$." If the type sets were not disjoint, we could still generalize the proposition. In a more general setting, of course, a type cannot necessarily be said to have a strategy associated with it, since the strategy may differ between players.

Proposition 4. *If for a game $G \subseteq \mathcal{G}$, the profile of actions (L_x, L_y) played by the prior infected set I is p-dominant for $CE^p(I)$, then for every $k \in \mathbb{N}$, in any equilibrium, if $t \in (BE^p)^k(I)$, then t plays L_i. Moreover, for any $t \in CE^p(I)$, t plays L_i in any equilibrium.*

Proof. We will show this by induction.

Suppose there is a type $t \in BE^p(I)$ that plays an action other than L_i with positive probability in some equilibrium. We know $t \notin I$, since L_i is strictly dominant for types in I. Thus, $\pi(I \mid t) > p$. Type t then believes with probability greater than p that her opponent will play action L_j, but since (L_x, L_y) is p-dominant for t, it would be better for t to play only L_i. Our assumption of equilibrium was thus contradictory. Thus, in any equilibrium, if $t \in BE^p(I)$, then t plays L_i.

Now assume that in any equilibrium, types in $(BE^p)^k(I)$ play L_i, but that there exists an equilibrium in which some type $t \in (BE^p)^{k+1}(I)$ plays an action other than L_i with positive probability. We cannot have $t \in (BE^p)^k(I)$, since this would violate our inductive hypothesis. Thus, $\pi((BE^p)^k(I) \mid t) > p$. Type t believes with probability greater than p that his opponent will play action L_j, but since $t \in CE^p(I)$, (L_x, L_y) is p-dominant for t, and so t could do strictly better by only playing L_i. This gives us a contradiction, so we conclude that in any equilibrium, if $t \in (BE^p)^{k+1}(I)$, then t plays L_i.

For every $t \in CE^p(I)$, $t \in (BE^p)^k(I)$ for some $k \in \mathbb{N}$. Thus, in any equilibrium, if $t \in CC^p(T)$, then for the corresponding player i, $\sigma_i(t) = L_i$. □

In Proposition 4, we specified that the action profile (L_x, L_y) was p-dominant for $CE^p(I)$. If the action profile is also p-dominant for, say, the entire set $\mathcal{T}$, this would not change the result: the set that was forced to play L_x or L_y stops growing if $CE^p(I)$ stops growing, not because (L_x, L_y) is not p-dominant for the entire set $\mathcal{T}$.

Proposition 4 shows that if there is a set I and an action pair (L_x, L_y) that is p-dominant for $CE^p(I)$, then if types in I, play L_i, then all types in the set $CE^p(I)$ will play L_i in any equilibrium. In this sense, if an information system P is (η, p)-contagious, then for any game with information system P such that there is a p-dominant action pair (L_x, L_y) for $CE^p(I)$, it is possible to infect a set of types I that occurs with small prior probability (less than η) such that a set of types $CE^p(I)$ that occurs with arbitrarily high prior probability (greater than $1-\eta$) will play L_x or L_y.

For the following proposition, confer with Proposition 3 from Morris [8] and Lemma 5.2 from Kajii and Morris [4].

Proposition 5. *If for a game G, there exists an action pair (α_x, α_y) that is strongly q-dominant for $CC^q(T)$, then there exists an equilibrium in which all types in $CC^q(T)$ play α_i.*

Proof. Suppose all types in $CC^q(T)$ play action α_i. We wish to show that no player would wish to deviate. Let $t \in CC^q(T)$. Then $\pi(CC^q(T) \mid t) \geq q$. Type t then believes with probability at least q that her opponent will play α_j. Since (α_x, α_y) is strongly q-dominant, it is strictly better for t to play α_i than any other action. Thus, no type $t \in CC^q(T)$ would want to deviate. There thus exists an equilibrium in which types $t \in CC^q(T)$ play an action α_x or α_y. □

Propositions 4 and 5 show sufficient conditions under which we can have coexistent equilibria in the sense that equilibria exist in which different types play different actions. This is related to our notion of (η, p)-contagiousness, since infection of all types occurs if and only if there are no coexistent equilibria. Since (η, p)-contagiousness is a notion of *partial* contagion, we are concerned primarily with the the size (prior probability) of the different sets. The following corollary provides a lower bound on how large one of the sets must be.

Corollary 2 (Coexistence)**.** *Let G be a game in which the pair of actions (L_x, L_y) played by the infected set I is p-dominant for $\mathcal{T}$ and there exists a separate action pair (α_1, α_2) that is strongly q-dominant for $CC^q(\bar{I})$. If $CC^q(\bar{I})$ is nonempty, then in some equilibrium, action pairs (L_x, L_y) and (α_x, α_y) are both played. More specifically, there exists an equilibrium in which types in $CE^p(I)$ play L_i, and types in $CC^q(\bar{I})$ play α_i.*

Proof. This follows immediately from Proposition 4 and Proposition 5. □

5 Properties of Information Systems

With a better understanding of what (η, p)-contagiousness means, we now examine properties of an information system that promote a high level of contagiousness (high p), as well as limits on how contagious an information system can be.

We define a labeling as $l : \mathbb{N} \to \mathcal{T}$ or $l : \mathbb{Z} \to \mathcal{T}$, which allows us to prove a useful and intuitive result. Define $T_{<K}$ ($T_{>K}$) as the set of types with label less than (greater than) K. Formally:

$$T_{<K}(l) := \{l(k) \in \mathcal{T} \mid k < K\}$$
$$T_{>K}(l) := \{l(k) \in \mathcal{T} \mid k > K\}.$$

Note that for any $k \geq 1$, $\pi(T_{<k}(l) \mid l(k)) + \pi(T_{>k}(l) \mid l(k)) = 1$.

Proposition 6. *An information system $P \in \mathcal{P}$ is (η, p)-contagious if there is a labeling l such that for some J, K, where $J > K$, $\pi(T_{>J}(l)) \leq \eta$, $\pi(T_{<K}(l)) \leq \eta$, one of the following holds:*

a.) for every k such that $J \geq k \geq K$, $\pi(T_{<k}(l) \mid l(k)) > p$

b.) for every k such that $J \geq k \geq K$, $\pi(T_{>k}(l) \mid l(k)) > p$,

Proof. We will prove this using induction.

Assume there is a labeling such that *a.*) is true. $\pi(T_{<K}(l) \mid l(K)) > p$, so $l(K) \in BE^p(T_{<K}(l))$. Now assume there is some k such that $J - K > k \geq 0$ and that for all j such that $k \geq j \geq 0$, $l(K+j) \in (BE^p)^{j+1}(T_{<K}(l))$. Now we wish to show $l(K+k+1) \in (BE^p)^{k+2}(T_{<K}(l))$. Because BE^p is monotonic,

$$l(K+j) \in (BE^p)^{j+1}(T_{<K}(l)) \Rightarrow l(K+j) \in (BE^p)^{j+m}(T_{<K}(l))$$

for any $m \geq 1$. Thus, our inductive hypothesis implies that

$$T_{<K}(l) \cup \{l(K), l(K+1), \ldots, l(K+k)\} \subseteq (BE^p)^{k+1}(T_{<K}(l))$$

since

$$T_{<K}(l) \subseteq BE^p(T_{<K}) \subseteq (BE^p)^{k+1}(T_{<K}(l))$$

and for any j such that $k \geq j \geq 0$,

$$l(K+j) \in (BE^p)^{j+1}(T_{<K}(l)) \subseteq (BE^p)^{k+1}(T_{<K}(l)).$$

We thus have,

$$\begin{aligned}\pi\left((BE^p)^{k+1}(T_{<K}(l)) \mid l(K+k+1)\right) &\geq \pi\left(T_{<K}(l) \cup \left(\bigcup_{i=K}^{K+k}\{l(i)\}\right) \mid l(K+k+1)\right)\\ &= \pi(T_{<K+k+1}(l) \mid l(K+k+1))\\ &> p,\end{aligned}$$

and so $l(K+k+1) \in (BE^p)^{k+2}(T_{<K})$.

Thus for any k such that $J \geq k \geq K$, $l(k) \in CE^p(T_{<K})$. All types are then in the set $CE^p(T_{<K})$ except possibly for some types in the set $T_{>J}(l)$, where $\pi(T_{>J}(l)) \leq \eta$. Thus, $\pi(T_{<K}) \leq \eta$ and $\pi(CE^p(T_{<K})) \geq 1-\eta$, so P is (η, p)-contagious.

Suppose now that there is a labeling such that *b.*) holds instead of *a.*). This part of the proof is analogous to the part above. $\pi(T_{>J}(l) \mid l(J)) > p$, so $l(J) \in BE^p(T_{>J}(l))$. Now assume that there is some k such that $J-K > k \geq 0$ and that for every $k \geq j \geq 0$, $l(K-j) \in (BE^p)^{j+1}(T_{>J}(l))$. For $k+1$, we have $\pi(T_{>J-k-1}(l) \mid l(J-k-1)) > p$. Thus,

$$\begin{aligned}\pi\left((BE^p)^{k+1}(T_{>J}(l)) \mid l(J-k-1)\right) &\geq \pi\left(T_{>J}(l) \cup \left(\bigcup_{i=J-k-1}^{J}\{l(i)\}\right) \mid l(J-k-1)\right)\\ &= \pi(T_{>J-k}(l) \mid l(J-k-1))\\ &> p,\end{aligned}$$

so we see $l(J-k-1) \in (BE^p)^{k+2}(T_{>J}(l))$.

For any k such that $J \geq k \geq K$, $l(k) \in CE^p(T_{>K}(l))$. All types are then in the set $CE^p(T_{>K}(l))$ except for some set of types $T_{<K}(l)$, where $\pi(T_{<K}(l)) \leq \eta$. □

Corollary 3. *An information system $P \subseteq \mathcal{P}$ is p-contagious if for every $\epsilon > 0$, there is a labeling l such that for some J, K, where $J > K$, $\pi(T_{>J}(l)) < \epsilon$, $\pi(T_{<K}(l)) < \epsilon$, one of the following holds:*

i.) for every k such that $J \geq k \geq K$, $\pi(T_{<k}(l) \mid l(k)) > p$

ii.) for every k such that $J \geq k \geq K$, $\pi(T_{>k}(l) \mid l(k)) > p$.

We are interested primarily in the sufficient conditions for contagion. We note though that the converses of Proposition 6 and Corollary 3 hold as well if we extend the definition of labeling to include labelings of the form $l : \mathbb{Z}^n \to \mathcal{T}$ for any n and $l : \mathbb{Z}^\Omega \to \mathcal{T}$.

5.1 Uniformity

The results in the last section relied on players' being able to think of an arbitrarily high number of iterations of the form I believe that my opponent believes that I believe ... that my opponent has such a type that L_i is a dominant action. Of course, we do not expect individuals to make such calculations. Proposition 7 and its corollary allow individuals to take a shortcut in determining the contagiousness of an information system if they know it is sufficiently uniform. We first define our notion of uniformity, which is a variant of that introduced by Morris [9].

Definition 4 ((η, δ)-Uniformity)**.** An information system $P \in \mathcal{P}$ is (η, δ)-uniform if there exists a labeling l with J, K, where $J > K$, $\pi(T_{<K}(l)) \leq \eta$, and $\pi(T_{>J}(l)) \leq \eta$, such that

$$\max_{k,k' \in \{K,\ldots,J\}} |\pi(T_{<k}(l) \mid l(k)) - \pi(T_{<k'}(l) \mid l(k'))| < \delta.$$

If a game is (η, δ)-uniform for every $\eta > 0$, then we say it is δ-uniform.

Proposition 7. *If an information system $P \in \mathcal{P}$ is (η, δ)-uniform, then it is (η, p)-contagious for some $p \geq \frac{1}{2} - \delta$.*

Proof. Since P is (η, δ)-uniform, there exist J, K, where $J > K$, $\pi(T_{<K}) \leq \eta$, and $\pi(T_{>J}) \leq \eta$, such that for any k, $J \geq k \geq K$,

$$\alpha - \delta < \pi(T_{<k}(l) \mid l(k)) < \alpha$$

for some α. Since $\pi(T_{>k}(l) \mid l(k)) = 1 - \pi(T_{<k}(l) \mid l(k))$, we get

$$1 - \alpha < \pi(T_{>k}(l) \mid l(k)) < 1 - \alpha + \delta.$$

We see $\max\{\alpha - \delta, 1 - \alpha\} \geq \frac{1}{2} - \delta$. This implies for all k such that $J \geq k \geq K$, either

$$\frac{1}{2} - \delta < \pi(T_{<k}(l) \mid l(k))$$

or

$$\frac{1}{2} - \delta < \pi(T_{>k}(l) \mid l(k)).$$

By Proposition 6, P is $(\eta, (\frac{1}{2} - \delta))$-contagious. □

Corollary 4. *If an information system $P \in \mathcal{P}$ is δ-uniform, then P is p-contagious for some $p \geq \frac{1}{2} - \delta$.*

Proof. The proof of Proposition 7 can be repeated for arbitrary $\epsilon > 0$, and we would get the same lower bound of $\frac{1}{2} - \delta$. By Corollary 3, P is p-contagious. □

Morris [9] proved a result akin to Proposition 7 for local interaction games, but it required low neighbor growth. We have focused on partial contagion, though we emphasize that if we required a different requirement for contagion, along the lines of

For every $\epsilon > 0$, there exists a set I such that $\pi(I) < \epsilon$ and $CE^p(I) = 1$,

we could use Proposition 8 to prove a more general version of Proposition 7. This follows because for information systems, the prior probability of an event is bounded by 1, which provides a constraint analogous to low neighbor growth.

5.2 Bound on Contagiousness

In this section we provide limits on how contagious an information system can be. Proposition 8 allows us to relate the prior probability η to the parameter p in (η, p)-contagiousness.

Proposition 8. *If $I \subseteq \mathcal{T}$, then $\pi((BE^p)^k(I)) < \sum_{j=0}^{k} \left(\frac{1-p}{p}\right)^j (\pi(I))$. Moreover, if $p < \frac{1}{2}$, then $\pi(CE^p(I)) < \frac{p}{2p-1}(\pi(I))$.*

We largely follow the proof of Oyama and Tercieux [12]. We use four lemmas in our proof. Lemma 5 is used for Lemma 6, which allows us to relate how large a set $BE^p(I)$ can grow as a function of I and the prior probability of types that are in $BE^p(I) \backslash I$, but interact with types in I. We then introduce some more notation that is fairly opaque, but provides a straightforward proof of the proposition using Lemma 7 and Lemma 8.

Lemma 5. *If $I \subseteq X_i$, and $T \subseteq BE^p(I)$, then $\mu(\mathcal{I}(T) \backslash \mathcal{I}(I)) < \frac{1-p}{p} \mu(\mathcal{I}(T) \cap \mathcal{I}(I))$.*

Proof. Denote the elements of T by $t_1, t_2, \cdots$. Note that by definition, for every t_i, $\mu(\mathcal{I}(I) \mid \mathcal{I}(t_i)) > p$.

$$\begin{aligned}
\mu(\mathcal{I}(I) \mid \mathcal{I}(T)) &= \frac{\mu(\mathcal{I}(I) \cap \mathcal{I}(T))}{\mu(\mathcal{I}(T))} & (1)\\
&= \frac{\mu(\mathcal{I}(I) \cap (\mathcal{I}(t_1) \cup \mathcal{I}(t_2) \cup \cdots))}{\mu(\mathcal{I}(T))} & (2)\\
&= \frac{\mu(\mathcal{I}(I) \cap \mathcal{I}(t_1)) + \mu(\mathcal{I}(I) \cap \mathcal{I}(t_2)) + \cdots}{\mu(\mathcal{I}(t_1) + \mu(\mathcal{I}(t_2) + \cdots} & (3)\\
&> p & (4)
\end{aligned}$$

Note that Equation 3 follows because for every $i \neq j$, $\mathcal{I}(t_i) \cap \mathcal{I}(t_j) = \emptyset$. Note further that we obtain $\mu(\overline{\mathcal{I}(I)} \mid \mathcal{I}(T)) < 1 - p$.

By Bayes' rule, $\mu(\mathcal{I}(I) \mid \mathcal{I}(T)) > p$ implies that

$$\mu(\mathcal{I}(I) \cap \mathcal{I}(T)) > p\mu(\mathcal{I}(T)).$$

Similarly, $\mu(\overline{\mathcal{I}(I)} \mid \mathcal{I}(T)) < 1 - p$ implies

$$\mu(\overline{\mathcal{I}(I)} \cap \mathcal{I}(T)) < (1-p)\mu(\mathcal{I}(T)).$$

We thus obtain

$$\mu(\overline{\mathcal{I}(I)} \cap \mathcal{I}(T)) < \frac{1-p}{p}\mu(\mathcal{I}(I) \cap \mathcal{I}(T)).$$

□

Lemma 6. *If $I \subseteq \mathcal{T}$ and $T \subseteq (BE^p(I)\backslash I)$, then $\mu(\mathcal{I}(T)\backslash\mathcal{I}(I)) < \frac{1-p}{p}\mu(\mathcal{I}(T) \cap \mathcal{I}(I))$.*

Proof. We can use the fact that $T = T_x \cup T_y$, where $T_i = T \cap X_i$, to obtain by Lemma 5 that

$$\begin{aligned}
\mu(\overline{\mathcal{I}(I)} \cap \mathcal{I}(T_x)) <& \frac{1-p}{p}\mu(\mathcal{I}(I) \cap \mathcal{I}(T_x)) \\
\mu(\overline{\mathcal{I}(I)} \cap \mathcal{I}(T_y)) <& \frac{1-p}{p}\mu(\mathcal{I}(I) \cap \mathcal{I}(T_y)).
\end{aligned}$$

Combining the two equations, we get

$$\begin{aligned}
&\mu(\overline{\mathcal{I}(I)} \cap \mathcal{I}(T_x)) + \mu(\overline{\mathcal{I}(I)} \cap \mathcal{I}(T_y)) \\
&\qquad < \frac{1-p}{p}\big(\mu(\mathcal{I}(I) \cap \mathcal{I}(T_x)) + \mu(\mathcal{I}(I) \cap \mathcal{I}(T_y))\big).
\end{aligned}$$

This is equivalent to

$$\begin{aligned}
&\mu\big((\overline{\mathcal{I}(I)} \cap \mathcal{I}(T_x)) \cup (\overline{\mathcal{I}(I)} \cap \mathcal{I}(T_y))\big) + \mu(\overline{\mathcal{I}(I)} \cap \mathcal{I}(T_x) \cap \mathcal{I}(T_y)) \\
&\qquad < \frac{1-p}{p}\Big(\mu\big((\mathcal{I}(I) \cap \mathcal{I}(T_x)) \cup (\mathcal{I}(I) \cap \mathcal{I}(T_y))\big) + \mu(\mathcal{I}(I) \cap \mathcal{I}(T_x) \cap \mathcal{I}(T_y))\Big).
\end{aligned}$$

This gives

$$\begin{aligned}
\mu(\overline{\mathcal{I}(I)} \cap \mathcal{I}(T)) <& \frac{1-p}{p}\mu(\mathcal{I}(I) \cap \mathcal{I}(T)) \\
&+ \Big(\frac{1-p}{p}\mu\big(\mathcal{I}(I) \cap \mathcal{I}(T_x) \cap \mathcal{I}(T_y)\big) - \mu\big(\overline{\mathcal{I}(I)} \cap \mathcal{I}(T_x) \cap \mathcal{I}(T_y)\big)\Big).
\end{aligned}$$

And since $\mu\big(\mathcal{I}(I) \cap \mathcal{I}(T_x) \cap \mathcal{I}(T_y)\big) = 0$, we get that

$$\mu(\overline{\mathcal{I}(I)} \cap \mathcal{I}(T)) < \frac{1-p}{p}\mu(\mathcal{I}(I) \cap \mathcal{I}(T)).$$

□

Now we introduce some more notation that is a bit opaque, but is very helpful in proving the proposition. For $I \subseteq \mathcal{T}$, denote $E^0 = I$ and $E^k = (BE^p)^k(I)$. Define

$$\begin{aligned}
B_K :=& \sum_{j=1}^{K} \mu(\mathcal{I}(E^j\backslash E^{j-1}) \cap \mathcal{I}(E^{j-1})) \\
C_K :=& \sum_{j=1}^{K} \mu(\mathcal{I}(E^j\backslash E^{j-1})\backslash\mathcal{I}(E^{j-1})),
\end{aligned}$$

and set $B_0, C_0 = 0$. Note that $C_K + \mu(\mathcal{I}(E^0)) = \mu(\mathcal{I}(E^K)) = \pi(BE^K(I))$. For example with E^1,

$$\begin{aligned}
C_1 + \mu(\mathcal{I}(E^0)) =& \mu(\mathcal{I}(E^1\backslash E^0)\backslash\mathcal{I}(E^0) + \mu(\mathcal{I}(E^0)) \\
=& \mu((\mathcal{I}(E^1\backslash E^0)\backslash\mathcal{I}(E^0)) \cup \mathcal{I}(E^0)) \\
=& \mu((\mathcal{I}(E^1\backslash E^0) \cup \mathcal{I}(E^0)) \cap (\overline{\mathcal{I}(E^0)} \cup \mathcal{I}(E^0))) \\
=& \mu(\mathcal{I}(E^1)).
\end{aligned}$$

If we compare each term in the sum for C_K with the corresponding term in the sum of B_K, since for every $k \geq 1$,

$$E^k \backslash E^{k-1} \subseteq BE^p(E^{k-1}) \backslash E^{k-1},$$

it directly follows from Lemma 6 that $C_K < \frac{1-p}{p} B_K$. This relationship will be used in Lemma 8.

Lemma 7. $B_K \leq C_{K-1} + \mu(\mathcal{I}(E^0))$.

Proof. Because, for example,

$$\big(\mathcal{I}(E^K \backslash E^{K-1}) \cap \mathcal{I}(E^{K-1}\big) \cap \big(\mathcal{I}(E^{K-1} \backslash E^{K-2}) \cap \mathcal{I}(E^{K-2})\big) = \emptyset,$$

we can rearrange the terms in the sum of B_K as follows:

$$\begin{aligned} B_K =& \mu(\mathcal{I}(E^K \backslash E^{K-1}) \cap \mathcal{I}(E^{K-1})) + \mu(\mathcal{I}(E^{K-1} \backslash E^{K-2}) \cap \mathcal{I}(E^{K-2})) \\ & + \cdots + \mu(\mathcal{I}(E^1 \backslash E^0) \cap \mathcal{I}(E^0)) \\ =& \mu\Big((\mathcal{I}(E^K \backslash E^{K-1}) \cap \mathcal{I}(E^{K-1})) \cup (\mathcal{I}(E^{K-1} \backslash E^{K-2}) \cap \mathcal{I}(E^{K-2})) \cup \cdots \\ & \cup (\mathcal{I}(E^1 \backslash E^0) \cap \mathcal{I}(E^0))\Big). \end{aligned}$$

And if we examine the argument of μ in the previous equation, we see

$$\Big(\big(\mathcal{I}(E^K \backslash E^{K-1}) \cap \mathcal{I}(E^{K-1}\big) \cup \cdots \cup \big(\mathcal{I}(E^1 \backslash E^0) \cap \mathcal{I}(E^0)\big)\Big) \subseteq \mathcal{I}(E^{K-1}).$$

Thus,

$$B_K \leq \mu(\mathcal{I}(E^{K-1})) = C_{K-1} + \mathcal{I}(E^0).$$

□

Lemma 8. $C_K < \frac{1-p}{p}\big(C_{K-1} + \mu(\mathcal{I}(E^0))\big)$.

Proof. By Lemma 7 and the fact that $C_K < \frac{1-p}{p} B_K$,

$$C_K < \frac{1-p}{p} B_K \leq \frac{1-p}{p}\big(C_{K-1} + \mu(\mathcal{I}(E^0)).$$

□

We can now prove Proposition 8: *If $I \subseteq \mathcal{T}$, then $\pi((BE^p)^k(I)) < \sum_{j=0}^{k} \left(\frac{1-p}{p}\right)^j \pi(I)$. Moreover, if $p > \frac{1}{2}$, then $\pi(CE^p(I)) < \frac{p}{2p-1}(\pi(I))$.*

Proof. The first part follows by recursively applying Lemma 8, and the facts that $\pi((BE^p)^k(I) = C_k + \mu(\mathcal{I}(E^0))$ and $\mu(\mathcal{I}(E^0)) = \pi(I)$. The second part follows from taking the limit of the first part as $k \to \infty$. □

Corollary 5. *If an information system $P \in \mathcal{P}$ is p-contagious, then $p \leq \frac{1}{2}$.*

Using Proposition 8, for an information system that is (η, p)-contagious, we can provide bounds on η and p as functions of each other with the following proposition.

Proposition 9. *If an information system $P \in \mathcal{P}$ is (η, p)-contagious and $p > \frac{1}{2}$, then $\eta > \frac{2p-1}{3p-1}$. If P is (γ, p)-contagious, then $p < \frac{1-\gamma}{2-3\gamma}$.*

Proof. If $p > \frac{1}{2}$, then for any $I \subseteq \mathcal{T}$ such that $\pi(I) \leq \eta$ and $\pi(CE^p(I)) \geq 1-\eta$, we must have by Proposition 8 that

$$1 - \eta \leq \pi(CE^p(I)) < \frac{p}{2p-1}\pi(I) \leq \frac{p}{2p-1}\eta.$$

This gives $1 - \eta(1 + \frac{p}{2p-1}) < 0$, which implies

$$\eta > \left(1 + \frac{p}{2p-1}\right)^{-1} = \frac{2p-1}{3p-1}.$$

Now assume P is (γ, p)-contagious. Then for some I,

$$\pi(CE^p(I)) \leq 1 - \eta < \frac{p}{2p-1}\eta \leq \frac{p}{2p-1}\pi(I).$$

We can rearrange the inner two inequalities to obtain $p < \frac{1-\eta}{2-3\eta}$. □

For information systems that are (η, δ)-uniform for low values of η and δ, we can thus provide a narrow range of how contagious the information system is. For example, if an information system is δ-uniform, then it is p-contagious for $\frac{1}{2} - \delta \leq p \leq \frac{1}{2}$. And if an information system is (η, δ)-uniform, then $\frac{1}{2} - \delta \leq p \leq \frac{1-\eta}{2-3\eta}$. For smaller and smaller η, these bounds become increasingly closer.

Acknowledgements

I would like to thank the Santa Fe Institute and the National Science Foundation for support, and Willemien Kets for her excellent mentorship.

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