# Modeling the degree distribution of a fractal transportation network with a minimum spanning tree graph

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## Abstract

We model the topology of experimental fractal particle agglomerates. We find that the degree distribution of minimum spanning trees on a random distribution of nodes and the degree distribution of diffusion limited aggregation (DLA) clusters are close to the experiment. DLA clusters have $23.8\% \pm 0.5\%$ degree one nodes, compared to $21.9\% \pm 0.8\%$ in the experiment, $55.0\% \pm 0.9\%$ degree two nodess compared to $56\% \pm 1.8\%$ in the experiment, and $20.1\% \pm 0.5\%$ degree three nodes compared to $21.8 \pm 0.8\%$ in the experiment. In contrast, trees grown with the propagation front model on a random distribution of nodes have 37% degree one nodes. In addition we find that the total path length of DLA structures correlates well with the experiment, whereas minimum spanning trees and trees grown with the propagation front model have a 20% larger total path length. We conclude that the topology of DLA clusters matches the topology of the experimental particle agglomerates best.

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Ramified transportation networks are ubiquitous in many fields [1–3], ranging from river networks in geological sciences, to blood vessel systems, neural nets, and genetic regulatory networks in biology, and highway systems, power grids, and the Internet in engineering. However there have been only a few laboratory experiments on the growth and decay of ramified networks, including the polymerization of organic molecules [4] and the growth of fractal particle agglomerates due to a high-voltage current [5]. The limiting patterns of these fractal particle agglomerates (see Fig. 1) have a box-counting dimension of $D_{bc} = 1.67$, minimize the resistance and the energy consumption [6, 7], and have no closed loops. They can be considered as am hardware implementation of simple neural nets, since their dynamics is consistent with Hebb's learning rule [8]. From the underlying physical equations, Dueweke derived that the energy consumption is a Lyapunov function for the dynamics [9]. Marani *et al* used a simplified version of the equations to estimate the fractal dimension [10]. However, Marani's patterns have many closed loops in contrast to the experimental structures and other models, including diffusion limited aggregation (DLA) [11], which are trees.

In a two-dimensional embedding space, DLA creates patterns with fractal dimension 1.7 [12], which matches the experimental box-counting dimension. Viennot and Vanniemus [13] state that patterns with the same fractal dimension may have different topologies. Thus fractal measures are not enough to distinguish between models and the experiment, and graph theory concepts and language can be used as well. Jun *et al* [5] found that the experimental patterns have very simple degree distributions which are independent of the total number of nodes: $21.9\% \pm 0.8\%$ of all nodes have one neighbor (degree 1), $21.6\% \pm 0.8\%$ have three neighbors (degree 3) and all other nodes have two neighbors (degree 2). Nodes of degree four or higher are extremely rare. Unfortunately, the underlying physical equations (Maxwell's equations for the electric field, Ohm's law for the electrical current, the Navier-Stokes equations to model electro-convection, and Newton's second law with a complicated drag term to model the motion of the particles) are too complex to derive degree distributions and other topological properties of the emerging patterns numerically or analytically. Recently several researchers have attempted to model the evolution of complex networks with graph theoretical methods [2, 3].

In this paper we create ramified patterns with graph theoretical methods [14–16] and diffusion limited aggregation [11] and use the degree distribution [5], the total path length [17], and the box-counting dimension [18] to determine which pattern matches the topology of the experimental patterns best.

The following algorithm is used to approximate the minimum spanning tree for a random set of nodes [16]. First, a square lattice $Z^2$ with grid size 1 of $N$ nodes within a circle of radius $S$ is created. The position of the nodes are $(x_i, y_i)$, where $i = 1, 2, \ldots, N$. Second, each node is moved by $\Delta x_{1,i}$ horizontally and by $\Delta y_{1,i}$ vertically, where $\Delta x_{1,i}$ and $\Delta y_{1,i}$ are uniformly distributed random numbers in the interval $[-0.5, 0.5]$. Third, two nodes are picked at random and connected with an edge if the following conditions are satisfied: *(i)* their distance is less than the cut-off distance $d_c$, and *(ii)* there is no path between them. This step is repeated, until no more edges can be created. Initially the cut-off distance is $d_c = 0.2$. Fourth, the cut of distance $d_c$ is increased by 0.2 and step three is repeated. The algorithm terminates when all nodes are connected. Figure 2 shows two typical minimum spanning tree graphs and the degree distribution versus the number of nodes. A least square fit of the percentages of each degree as a function of the number of nodes is consistent with the hypothesis that they are independent of the number of nodes. $D_1 = 25.3\% \pm 0.3\%$ are of degree 1, $D_2 = 52.7\% \pm 0.6\%$ are of degree 2, $D_3 = 20.4\% \pm 0.4\%$ are of degree 3, and $D_4 = 1.6\% \pm 0.1\%$ are of degree 4. These values are close to experimentally observed degree distributions. If the nodes are seeded on a triangular lattice there are slightly more nodes of degree 1 and degree 3: $D_1 = 26.4\% \pm 0.4\%$, $D_2 = 50.6\% \pm 0.7\%$, $D_3 = 21.3\% \pm 0.4\%$, $D_4 = 1.6\% \pm 0.1\%$.

A procedure similar to Eden's algorithm [14, 15], the propagation front model, was also used. First, a distribution of nodes within a circle is created, just as in the minimum spanning tree model. Second, one of the leftmost nodes is chosen and "grounded". Third, a random non-grounded node is picked, an edge is created between this node and the closest grounded node within cut-off distance $d_c = 1.5$, if there is any, and the node is grounded. This step is repeated, until no more edges can be created. For cut-off distances $d_c$ less than 1.5 the number of isolated nodes (degree 0) increases sharply, and for $d_c = 1$ almost all nodes are isolated. For $d_c >= 1.5$ isolated nodes are very rare. Figure 3 shows typical propagation front graphs and the degree distribution versus the number of nodes. A least square fit of the percentages of each degree as a function of the number of nodes is consistent with the hypothesis that they are independent of the number of nodes, where $D_1 = 39.8\% \pm 0.6\%$, $D_2 = 31.5\% \pm 0.9\%$, $D_3 = 20.2\% \pm 0.7\%$, and $D_4 = 7.3\% \pm 0.4\%$. These degree distributions are qualitatively different from the experiment.

The following procedure is used to create DLA clusters [11]. First, there is only one node in the center. Second, a "walker" is randomly placed on a circle of radius 100 about the center. The walker moves along a random path determined by $W_{n+1} = W_n + (r_1, r_2)$ where $r_1$ and $r_2$ are uniformly distributed random numbers with $-0.5 < r_1 < 0.5$ and $-0.5 < r_2 < 0.5$, and $n = 1, 2, \ldots$. If the distance between the walker and a node is less than 2, then the walker turns into a node and the two nodes are connected with an edge. If the distance between the walker and the center exceeds 200, then the walker is deleted. Step two is repeated until $N$ nodes are created. Figure 4 shows typical DLA clusters and the degree distribution versus the number of nodes. A least square fit of the percentages of each degree as a function of the number of nodes is consistent with the hypothesis that they are independent of the number of nodes, where $D_1 = 23.8\% \pm 0.5\%$, $D_2 = 55.0\% \pm 0.9\%$, $D_3 = 20.1\% \pm 0.5\%$, and $D_4 = 1.1\% \pm 0.2\%$. These degree distributions are closest to the experimental values.

Since the degree distributions of the minimum spanning tree models and the DLA clusters are both close to the experimental values we use other graph theoretical measures to distinguish between the models. First, we determined the box counting dimension of the minimum spanning trees in the range $100 < N < 650$ and find that it is $D_{bc} = 1.75$ and independent of the number of nodes. This value is close to the experimental value and close to the box counting dimension of DLA clusters. Finally we determine the average value of the total path length [17]. The path length of a degree-one node is the number of nodes between the degree-one node and the root. The total path length is the sum of all path lengths of all degree-one nodes. As there is no obvious root in this case, the center of the graph is chosen to be the root. We find that the average value of the total path length of DLA clusters is about 20% less than the average value of total path length of minimum spanning tree graphs and graphs genenerated with the propagation front model. The average value of the total path length of the DLA clusters correlates well with the experimental values.

In summary, the degree distributions and the fractal dimensions of DLA clusters and minimum spanning tree graphs are independent of the number of nodes and close to the experimental values. The total path length of DLA clusters depends on the number of nodes, but agrees well with the experiment, whereas the total path length of the other models is about 20% higher. We conclude that the DLA clusters are the best model for the stationary states of fractal particle agglomerates in a high-voltage current.

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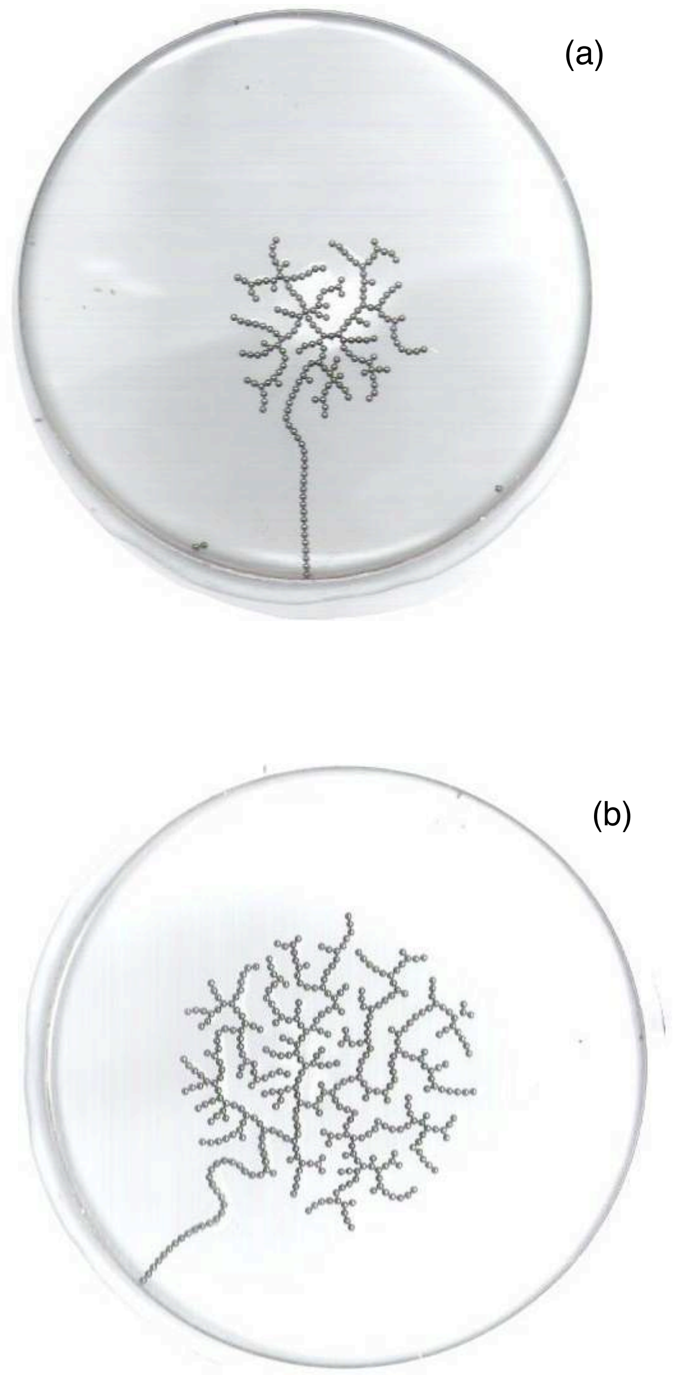


FIG. 1: Photographs of ramified particle aggregates in a high voltage current [5] with 193 particles (a) and with 392 particles (b).

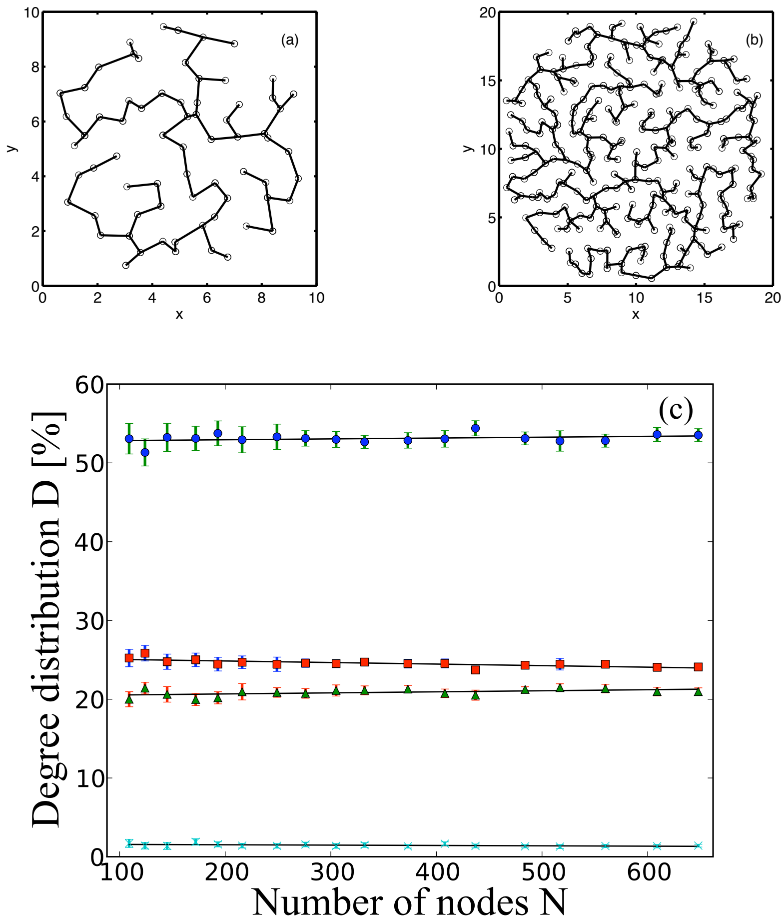


FIG. 2: Typical minimum spanning trees with 69 nodes (a) and 305 nodes (b), and the average percentage of nodes $D$ with degree 1 (square), degree 2 (circle), degree 3 (triangle) and degree 4 (x) versus the number of nodes $N$ (c). The error bars indicate the standard error for a set of 20 graphs each. A fit of the data (continuous lines) suggests that the degree distribution does not depend on the number of nodes.

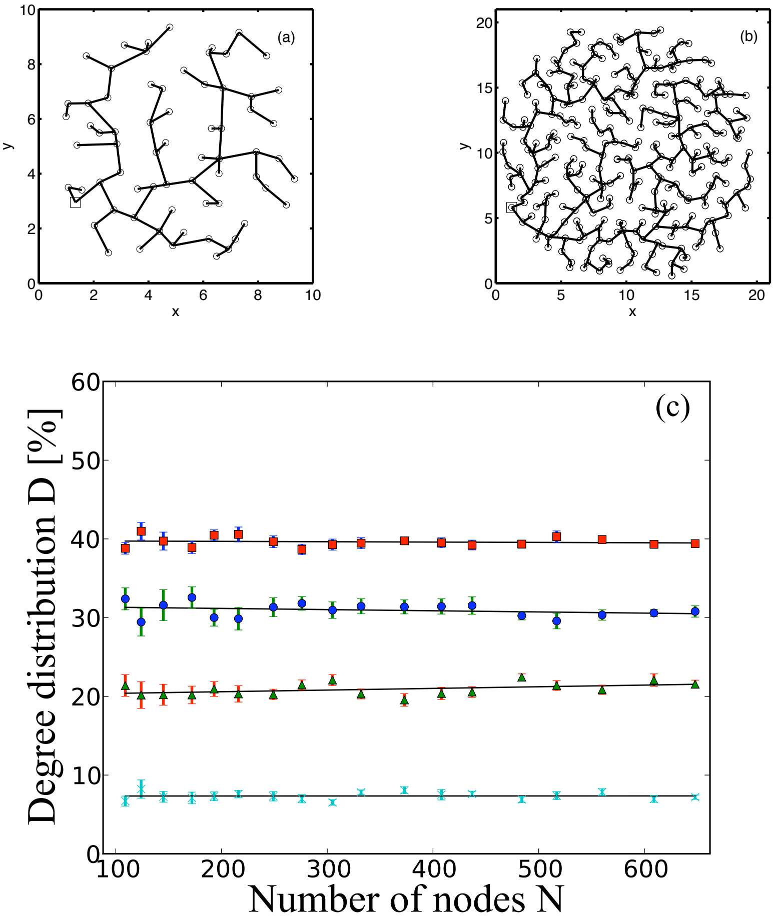


FIG. 3: A propagation front graph with 69 nodes (a) and 305 nodes (b) and the average percentage of nodes $D$ with degree 1 (square), degree 2 (circle), degree 3 (triangle) and degree 4 (x) versus the number of nodes $N$ (c). The error bars indicate the standard error for a set of 10 graphs each. A fit of the data (continuous lines) suggests that the degree distribution does not depend on the number of nodes. Initially only one node inside the square is grounded.

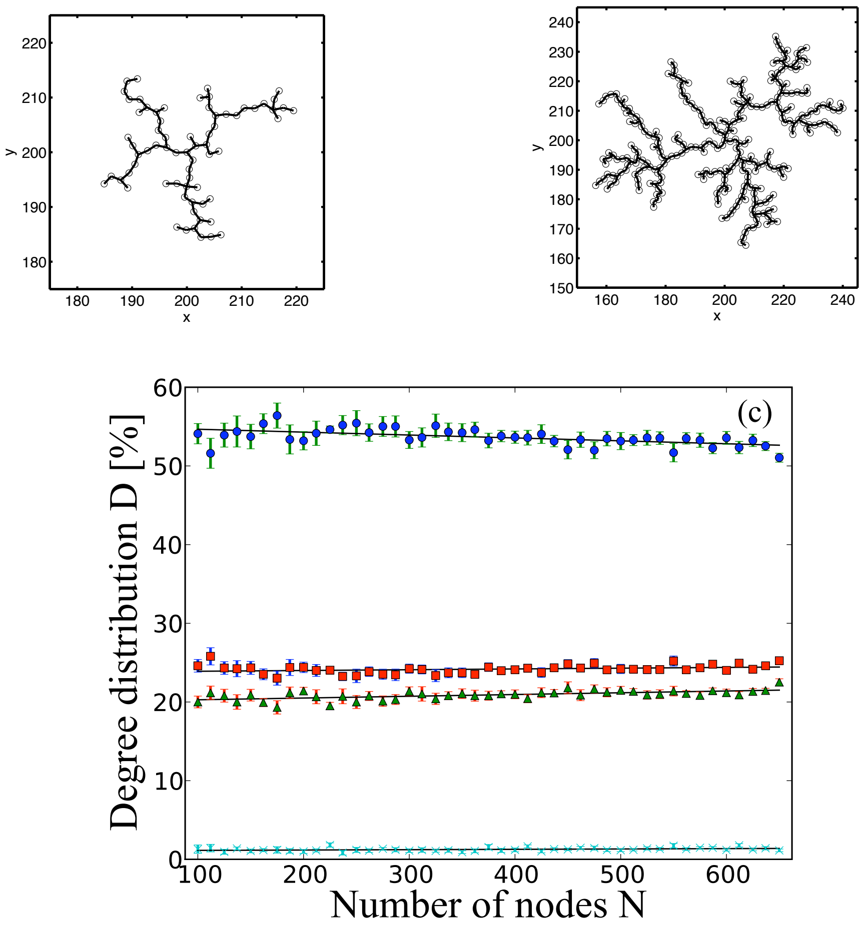


FIG. 4: Diffusion limited aggregation graph of size $N = 69$ (a) and of size $N = 305$ (b), and the average percentage of nodes $D$ with degree 1 (square), degree 2 (circle), degree 3 (triangle) and degree 4 (x) versus the number of nodes $N$ (c). The error bars indicate the standard error for a set of 10 graphs each. A fit of the data (continuous lines) suggests that the degree distribution does not depend on the number of nodes.