Bubbles and Crashes in a Heterogeneous-Agent Financial Market Model

Anirudh Jayanti ∗ Cristopher Moore †

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Abstract

We construct a heterogeneous-agent model of a single-asset financial market in which agents form demand for the asset based on price forecasts. Agents can change their forecasts over time in response to market conditions (in particular, price volatility and previous prices). We simulate this model and observe whether it produces the expected behavior of empirical financial time series.

Model

This is a discrete-time temporary equilibrium model. Much of the framework is based on that of Follmer, Horst and Kirman (2004). The sequence of events is as follows: traders make price forecasts, decide on a level of excess demand (with actual price unspecified), then equilibrium price is calculated based on the market-clearing condition of zero excess demand. The process repeats. The model shown algebraically is:

\[ e_t^a(p) = c^a(S_t^a - \log p) + \eta_t^a \]  

(1)

where \( e_t^a \) represents the excess demand of each agent \( a \in A \) at time \( t \), \( c^a \) is a time-invariant random parameter for each agent, \( S_t^a \) is the log price forecast of each agent at time \( t \), \( p \) is some proposed price and \( \eta_t^a \) is a random parameter that can be thought of as each agent’s liquidity demand at each time step. The market-clearing condition is:

∗Department of Mathematics and Economics, Dartmouth College, Hanover, NH 03755
†Resident Faculty, Santa Fe Institute, Santa Fe, NM 87501
\[ \sum_{a \in A} e_t^a(p) = 0 \]  

(2)

We substitute (1) into (2) to obtain the dynamics of the system:

\[ S_t := \log P_t = \frac{1}{c} \sum_{a \in A} c_t^a \hat{S}_t^a + \eta_t \]  

(3)

where

\[ c = \sum_{a \in A} c^a \]  

(4)

and

\[ \eta_t = \frac{1}{c} \sum_{a \in A} \eta_t^a. \]  

(5)

The \( \frac{1}{c} \) term in (5) was added for analytical convenience when deriving (3). Thus far, we are in the situation of Follmer et al (2004). We choose a different forecasting function based on endogenously-varying weights of fundamentalists and chartists. The forecasting rule \( \hat{S}_t^a \) is specified by:

\[ \hat{S}_t^a = S_{t-1} + \alpha \left( 1 - \frac{1}{1 + \sigma^2} \right) (F_a - S_{t-1}) + \beta \left( \frac{1}{1 + \sigma^2} \right) (S_{t-1} - S_{t-2}) \]  

(6)

where \( \sigma^2 \) represents price volatility and \( F_a \) is agent \( a \)'s opinion of the asset’s fundamental price. We measure price volatility by taking the variance of the set of ten previous prices, with a lag of ten periods (i.e. in time \( t \), the price volatility is calculated on the prices from \( t-20 \) to \( t-11 \)). The parameters \( \alpha \) and \( \beta \) are, respectively, measures of how quickly the agent believes prices will converge to fundamentals and how strongly he wishes to extrapolate from the previous trend.

Thus, the agent’s forecast is a combination of “fundamentalist” and “chartist” views, with fundamentalism given a weight depending on market conditions. (Alternatively, we could think of agents as being either strictly fundamentalist or strictly chartist and separate the forecast function, but this would simply change the number of agents in the model). In particular, in times of relative instability, agents will weight their fundamental opinions relatively more, whereas in times of stability, they will weight fundamentals relatively less. This replicates the principle of fundamentals mattering more “at a distance,” but it also weighs fundamentalist opinions more when prices are fluctuating chaotically, even if the fluctuations...
aren’t detached too much from the fundamental. In such a situation, chartists would not do well because there is no trend to extrapolate from. Thus, this model is even more general than previous ones because it accounts for more cases.

**Results**

In this section, we present the results from our simulations of the model. First, we specify our definitions of certain model parameters:

<table>
<thead>
<tr>
<th>$c^a$</th>
<th>$F^a$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\eta_t^e$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N(1, 0.5)$</td>
<td>$U(-0.5, 0.5)$</td>
<td>$U(0, 1)$</td>
<td>$N(1, 0.5)$</td>
<td>$N(0, 1)$</td>
</tr>
</tbody>
</table>

In all of the simulations described below, we use 100 agents. In Figure 1, we show a typical price plot after running the simulation for 1000 time steps. In this and all following sample plots, we discard the first 1000 time steps in order to “forget” initial conditions.
Notes: The plot shows the price series resulting from running the simulation for 1000 time steps. Bubbles and crashes appear.

Bubbles and crashes occur periodically, between periods of relative stability. One feature of the model worth emphasizing here is that due to our use of volatility as a control parameter, we do not need to place bounds on any of our parameters. The model does not perpetually display large fluctuations. In the plot below, we see that price spikes cause a decrease in the proportion of chartists in the economy (with a lag, by design).
Notes: This figure simultaneously plots the price series and the proportion of chartists in the economy over 1000 time steps. The proportion of chartists decreases when prices spike in either direction. The proportion is high during relatively stable periods.

Thus, the model works as it should. In the next figure, we plot price and volatility simultaneously:
Notes: This figure simultaneously plots the price series and price volatility, as defined in the previous section, in the economy over 1000 time steps. Volatility spikes when prices become unstable, by design. The volatility is mean-reverting.

The volatility tracks the price closely (again, by design) and is mean-reverting; the process does not explode to infinity. Next, we run the simulation for one million time steps. Figure 4 shows the resulting histogram, with a kernel density estimate and normal distribution with the same mean and variance.
Notes: Figure shows histogram of simulated price data (green) with a kernel density estimate (red) of the pdf. The distribution has fat tails compared to the normal distribution (blue) of the same mean and variance.

The kernel density estimate is more sharply peaked than the normal distribution at the center, implying that it has fatter tails than the normal distribution at the edges. To see this more clearly, we show the quantile-quantile plot in Figure 5 below:
Notes: Figure shows quantile-quantile plot of price distribution compared to a theoretical best-fit normal distribution (green line). The price distribution diverges significantly from the normal distribution at the edges, signifying fat tails.

Thus, the model displays fat tails in the price distribution, agreeing with a well-established stylized fact of financial time series. Next, we look at the tails of the price distribution on a log scale. Figure 6 shows that the tails decay exponentially, which is another well-documented feature of financial time series.

\(^1\)The return distribution also has fat tails, but there is not enough variance in the returns in this model to produce a meaningful distribution (in particular, both the kernel density estimate and normal distribution peak above 1).
Notes: Price distribution on log scale; tails show exponential decay. Empirical studies find this property in financial time series.

Thus, our model displays most of the empirical regularities observed in financial time series.

Conclusion

We find that our heterogeneous-agent model of the stock market agrees with many of the established stylized facts of financial time series. In particular, the price distribution has fat tails which decay exponentially. There is excess volatility in prices and returns (they can diverge significantly from fundamentals) and the volatility is mean-reverting. This is done without placing bounds on any of the parameters. Using price volatility as a control parameter is an effective self-correction mechanism in this type of model. Future work could generalize this model to allow for changing fundamental beliefs.