Toy Models of Interconnected Networks of Resources and Consumers

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Abstract

Real networks are not isolated, but are interlinked with and dependent on other real networks. This paper presents a set of tools for studying multiple interconnected networks. Utilizing simulation, a controlled environment can be created in which to understand multinetwork dynamics at small and large scales. The paper uses simulation to look at interconnected networks of resources and consumers. The paper examines when the removal of any single node in the resource network has the largest impact on the consumer network. The paper finds that it's import to focus on the network layer that connects the the separate networks of resources and consumers. In a controlled setting, the paper finds that increasing network density of the resource-consumer layer has a positive impact on minimizing negative consumer impact. Increasing Shannon's diversity index on outgoing degree in the resource-consumer layer minimizes worst case negative impact. Increasing Shannon's diversity on indegree maximizes negative impact. The paper also finds that separate networks are more "attuned" to one another under conditions of higher utilization. There is a wide avenue of future opportunities for further research in this new field of burgeoning interest in the social networks literature.

1 Introduction

Real networks are not isolated, but are interlinked with and dependent on other real networks (Radicchi and Arenas, 2013, Buldyrev et al., 2010). Examples of coupled networks are: the communication and power grid system, various transportation systems and energy supply networks. Analyses show that results from coupled networks differ from results based on single networks (Buldyrev et al., 2010). While studies performed on isolated scale-free networks have been shown to be robust against random failures of nodes and edges, the same is not true for interdependent scale-free networks that are more vulnerable to similar failures (Radicchi and Arenas, 2013). Failures in nodes and edges of one network cascade to the dependent networks, and as the functionality of the infrastructure depends on the functionality on the other networks, such failures can lead to a breakdown of the whole systems. Therefore studying the interdependencies of networks is important, because it may enable us to understand real networks better and to make their infrastructures more efficient and robust to failures.

Most real networks are scale-free (e.g. internet, WWW, social networks, aviation network, and networks in biology) which mean they follow a power-law form in their degree distribution (Gao et al., 2011). An example of a scale-free network is the air transportation network. Due to the scale-free property, the aviation network may be vulnerable to targeted attacks, but less vulnerable to random attacks. The aviation network, which is a consumer network, is depended on the functionality of the resource network, which is the oil supply network that fuels the aircrafts.

In the present study, we designed a toy-model (simplified model of real networks) inspired by the interdependence between a consumer (aviation) and a resource (oil) network. For example, A Boeing 747 uses 5 gallons of fuel per mile (www.Boeing.com), which corresponds to 8.4 miles travelled per barrel of oil. In the case of the aviation and the oil network, one can imagine how a decrease in oil supply to a country (e.g. in a war conflict) may cascade to a decrease in fuel supply to airports and flights. Another scenario that is likely to happen in the future is an increase in air traffic e.g. in Asian and African countries, which would result in a change in the infrastructure of the consumer and resource network.

Here we introduce and present a framework for analyzing and understanding interdependencies of and organizational principles of these network via a toy model. Questions we would like to investigate are: 1) under what conditions do changes in one network induce changes in a dependent network? 2) Can small changes in one network induce large changes in another network? A toy model inspired by the aviation (consumer network) and the oil flow (resource network) network was designed to answer the above questions.

We hypothesize that the structure of the intermediate layer that connects two coupled networks has important implications for the functioning of the network. When links going into and out of this intermediate layer are concentrated on a small number of nodes, we anticipate that the worst case consequences of node removal increase. To assess concentration, we utilize entropy measures for indegree and outdegree of nodes in the intermediate layer.

2 Methods

2.1 Defining an interconnected/multiplex graph

An interconnected graph can be defined as the union of graphs $G = G_1 \cup G_2 \cup ... = (V_1 \cup V_2 \cup ..., E_1 \cup E_2 \cup, ...)$, where E_i is a set of weighed edges representing a kind of relationship of vertices on set V_i. Further, a multiplex graph ($V_1 \equiv V_2 \equiv ...$) can be

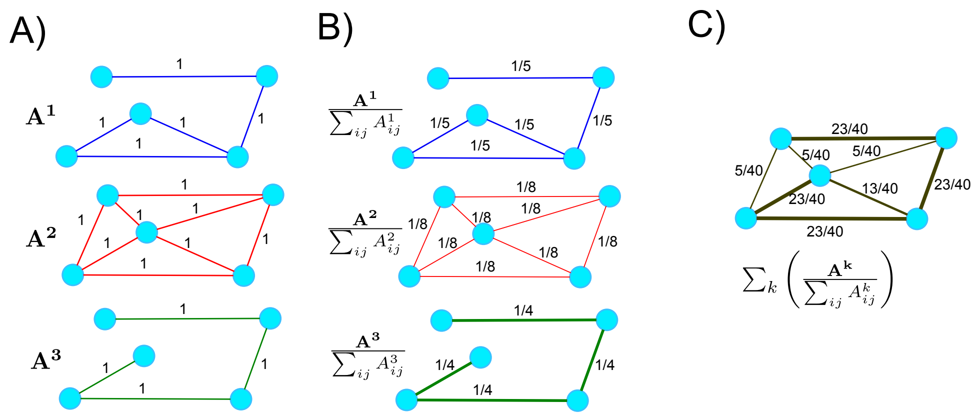

Figure 1: Process of normalizing layers in a multiplex network. **A)** Initially we have the same set of nodes with three different kind of edges. Each edge relationship is represented as a different adjacency matrix $\mathbf{A^k}$. **B)** Each edge relationship is normalized and the normalization is represented as $\mathbf{A^k}/\sum_{ij} A^k_{ij}$. **C)** Finally, equation (1) is applied with $w_1 = w_2 = w_3 = 1$.

represented as $G = (V, 0E_1 \cup E_2 \cup, ...)$. For example we can have that V represent0 the cities of a country, and E_1, E_2, and E_3 the number of people that travels across cities by plane, train and bus, respectively.

Each graph in the interconnected/multiplex network can be represented with the adjacency matrix $\mathbf{A^k}$, such that all of them have the same size and a row/columns represent the same node across all the matrices. Therefore empty rows/columns need to be included for the case of interconnected networks

2.2 Normalizing the interconnected/multiplex graph

We want to normalize all the edge weights, such that we can have a single adjacency matrix $\mathbf{B}$ that will represent the interconnected/multiplex graph. The elements of this matrix are defined as:

$$B_{ij} = \sum_k \left(w_k * \frac{A^k_{ij}}{\sum_{mn} A^k_{mn}} \right), \tag{1}$$

such that $w_k <= 1$ quantify how important is the interconnected network (or multiplex layer) in the graph. These values need to be assigned according to external information. For example in the case of a transportation multiplex network composed of fly, train and bus layers, the weights can be assigned according to the total flux of users. Another example can be different snapshots in time of the same graph. In this case, we may assign the values according to the rule $w_t < w_{t+1}$. Figure 1 shows an small example.

2.3 Applications

This way of normalizing the graph can be used to apply any kind of metric directly to a multiplex or interconnected network. Further, we can variate the weights in Equation (1) to study how measures as centrality, modularity,etc depend on the different layers of the network.

2.4 Toy Model

The toy model of our multiplex network consists of three layers that we can define as the resource layer network (oil network), the consumption layer network (fly network), and the intermediate layer that connects the oil and fly network.

This three-network structure is built from the bottom-up form resource to intermediate to consumer layers. Constraints are modeled into the networks layer by layer so that resource carriers cannot trade resources they don't possess with other resource carriers, so that consumers cannot fly to destinations that they don't obtain the requisite resources for.

2.5 Objectives

The main reason of this toy model is that we can create synthetic data that will allow us to study the dynamics of this kind of networks with the final goal of integrating real data. Another reason of using synthetic data first is that the real data may contain things that are not realistic (nodes that are sinks or sources). Ideally in the consumption network, any node can be a sink or source because that will mean that planes are created or destroyed from nowhere.

2.5.1 Time line

The time line for the constructing this toy model will be:

- Construct the resource layer network synthetically and test it using different degree distributions and random assignments for the required parameters (see below).
- Integrate the consumption layer network to the toy model ensuring that constraints are met. that is free of inconsistencies (nodes that are sinks or sources).
- Study the dynamics of the toy model under different types of disturbance.
- From the real oil network data we will try to infer what is the degree distribution and random distribution of the production, consumption, etc. And we will use this values to further study the oil network layer dynamics.
- Infer the structure of real world networks like that of oil and flight and create stylized model data to reflect this structure. Simulate network dynamics in this stylized setting.

3 Network generation

3.1 Resource layer network

We define a resource layer network as a weighted digraph $G_R = (V, E)$ that represents the flux of resources across different nodes, where the set V represent a set of nodes and E the set of edges. An edge $i \to j$ exist when node i exports $w_{ij} > 0$ resources to node j. A node can import and export to the same node. Further, at each time t each node i produces and consumes an amount $p_i(t)$ and $c_i(t)$ of resources, respectively.

Finally, each year the amount of reserves for each node is updated according to the following rule:

$$
\begin{aligned}
r_i(t+1) &= \text{current reserves} + \text{production} - \text{consumption} + \text{imports} - \text{exports} \\
&= r_i(t) + p_i(t) - c_i(t) + \sum_j w_{ji} - \sum_j w_{ij}
\end{aligned}
$$

3.2 Intermediate layer network

We define the intermediate (or resource-consumer) layer as edges that link resource carriers to consumers. The link is one way from resource carriers to consumers, and weights represent the amount of flux from resource carrier to consumer.

3.3 Consumer layer network

We define a consumer layer network as a weighted digraph $G_C = (V, E)$ that represents the resource-mediated relationship between different consumers. Edges between consumers are directed and require the a dependency on resources where the resources are not renewable or transferrable after use.

3.4 Simulation

3.4.1 Initialization

The simulation starts with a random configuration of:

- Random structure of graph G_R. The graph starts with $|V| = N$, with a degree distribution either from Erdos-Renyi or scale free random networks.
- Each link in the graph will have a weight randomly assigned in the range $w_{ij} \in \mathcal{U}(w_{min}, w_{max})$
- Each node will start with random assignments for its reserve, production and consumption at $t = 0$. That is we will randomly create $r_i(0) \in \mathcal{U}(r_{min}, r_{max})$, $p_i(0) \in \mathcal{U}(p_{min}, p_{max})$, $c_i(0) \in \mathcal{U}(c_{min}, c_{max})$. Further, for this part of the simulation (only the resource layer), we will consider that $p_i(t+1) = p_i(t)$ and $c_i(t+1) = c_i(t)$. In other words, the production and consumption of each node is constant during the entire simulation.

Ultimately our goal is to use a random uniform distribution for the initial assignments of the required quantities for the simulation. In future, the distribution of resources will model the actual distribution of oil, so that power law or normal distributions may be used.

For this initial phase, we will commence with a uniform assignment of the same amount of resources to each resource carrier in terms of production, and assume no reserves. This will help make our models more tractable. Once dynamics under this configuration are understood, we can alter the "initial conditions" that inform our simulations.

3.4.2 Simulation step

At each simulation time t we perform the following steps:

- Node selection: A node i is selected (either randomly or by a deterministic process).
- Node deletion: The node i is deleted. This will cause that its export and import links to be deleted.
- Graph rewiring: The deletion of node i will cause the decrease of resources in some of the nodes. However, we only need to focus for the nodes in which $r_j(t+1) \leq 0$. Two possibilities exist:
  - Delete some export links of node j randomly until $r_j(t+1) \geq 0$
  - If the last step is not possible delete node j
- Repeat last step as long as nodes with negative reserves exist.
- Re-add deleted node and its links, and commence simulation again.

For this paper, we will generate all valid networks where there are two consumers and two resource carriers. We will then generate all valid networks where there are three

4 Hypotheses and Results

4.1 Hypotheses

Hypothesis 1. *The higher the density of the resource-consumer layer, the lower the impact of a worst case impact from any single node removal.*

The intuition behind the first hypothesis regards redundancy. The more dense that the resource consumer layer is, the more opportunity for redundancy to consumers. This minimizes the impact of any single node's removal.

Hypothesis 2. *As the Shannon entropy increases from zero for the outdegree of the resource-consumer layer, the lower the impact of a worst case impact from any single node removal.*

The second hypothesis relates to the concentration of outgoing resources from resource carriers to consumers. The more that resources flow out in concentrated ways, the easier it is to knock out a node and thereby create large impacts. Calculating Shannon's diversity index provides a measure of concentration, and higher diversity equates to lower maximum loss.

Hypothesis 3. *As the Shannon entropy increases from zero for the indegree of the resource-consumer layer, the higher the impact of a worst case impact from any single node removal.*

For the last hypothesis, concentration of indegree is another way of saying that consumers have redundant sources of resources to utilize. Therefore, higher Shannon diversity scores regarding indegree should translate to higher maximum impact when a single resource node is removed. Of course, consumers that get horde resource links (and also consume them) might cause large systemic issues if removed. The issues of consumer removal is a subject for subsequent research.

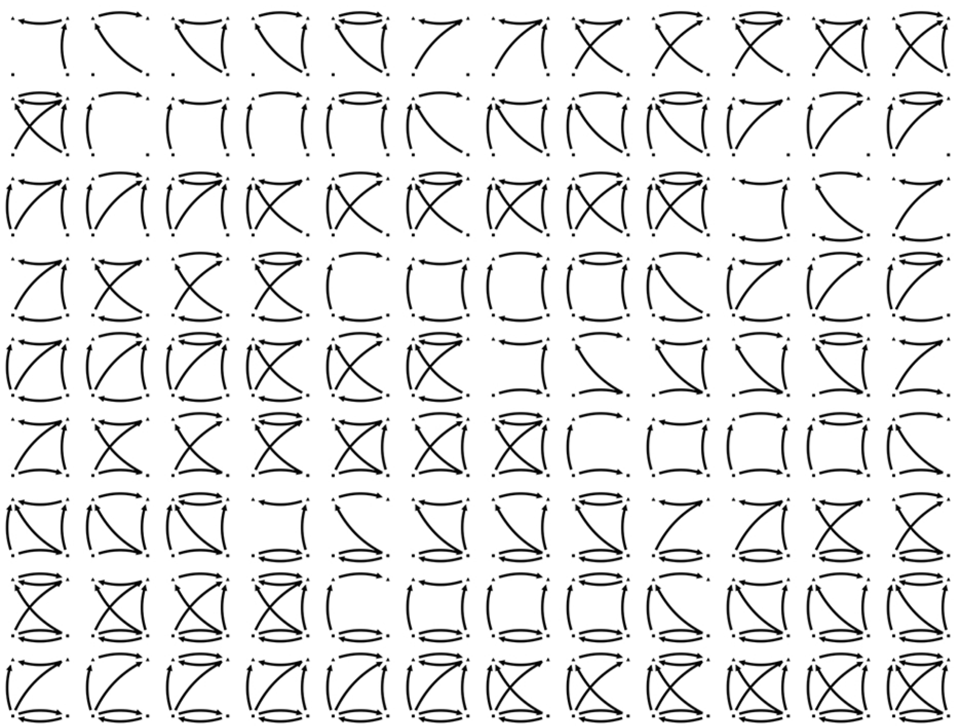

Figure 2: All possible two by two graphs. For each graph, the bottom row holds nodes that are resource carriers, and the top row holds consumers.

4.2 Generation of network models

Figure 2 shows the set of 108 valid graphs that are possible when each resource carrier is assigned a production value of 2 units, and all links are of unit weight. Each resource node is assigned a production value of n, being the number of consumer nodes. This means that, independent of imports from others, a resource carrier can on its own provide resources to all consumers. In our simplified model, resource carriers can only provide a single unit of resource to consumers.

A similar image for all 3x3 proved time prohibitive, as the image would consists of 809,664 individual graphs.

4.3 Results

To test our hypotheses, we utilize simple linear regression techniques. Table 1 presents regression tables obtained from our experiments with all 2x2 graphs. Our dependent variable is the lowest resulting number of consumer-to-consumer links remaining after any single resource node is removed from the network. From Table 1 Column 1, we find evidence that supports all three hypotheses. However, there is still a possibility of missing variable bias. This question could be alleviated by accounting for more of our data's variance. The inclusion of an indicator for "singleton consumers" increases r-squared to above 80%. A singleton consumer is one that has a degree of one on the resource-consumer layer, while having an outdegree of 1 in the consumer-consumer layer. Singleton consumer nodes can suffer greatly from the deletion of the single resource node. In graphs with only 1 consumer to consumer link, this can have a considerable negative impact. For the 2x2 graph scenario, 80% of graphs have only one consumer-consumer link. 25% of graphs have both singleton consumers and only one consumer-consumer link.

Table 1: Maximum negative impact on consumer-to-consumer links based on r-c network structure in a 2-resource by 2-consumer layer

	(1)	(2)
	max impact of node removal	
Shannon index for r-c outdegree	0.540***	0.358***
	(3.50)	(4.01)
Shannon index for r-c indegree	-0.787***	-0.453***
	(-4.67)	(-4.58)
r-c link density	0.425***	0.169***
	(5.88)	(3.76)
presence of 1-degree in/out consumers		-0.739***
		(-14.69)
Constant	-0.368**	0.513***
	(-3.18)	(5.74)
Observations	108	108
R^2	0.498	0.838

t statistics in parentheses

* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

To further validate our hypotheses, we conduct tests on all possible 3x3 networks. There are 809,664 graphs in this set. Table 2 presents regression models, this time excluding the control variable regarding singleton consumers, as singleton consumers are a collinear term with our entropy measures for this larger data set. Table 2 Column 1 shows the regression model run against all graphs. R-squared here is at 32%, with all hypotheses supported in the model. After noticing that model strength improved when restricting data to those of higher rc network density, Columns 2 and 3 were added to Table 2, where r-c density is restricted to above 60% and above 80% respectively. Here we also find that the the variance explained goes up to 58% and 70% with subsequent models, and that the impact of shannon entropy for both indegree and outdegree increases manyfold. This gives evidence that our hypotheses hold stronger weight in settings of higher r-c density. Why is this?

Figure 3 provides a hint of an answer. The figure shows that Shannon diversity scores of zero are found only when there are 3 or fewer c-c links. Indeed the range of possible Shannon scores is wider in these cases. A zero Shannon score indicates full concentration of outgoing links from one resource carrier to consumers. By definition of this experiment, one resource carrier can only link to a consumer with a unit weight. Since there a maximum of three consumer nodes, full outgoing concentration can only occur in scenarios with 3 or fewer c-c links.

The figure also suggests the idea that networks that aren't fully utilized may be harder to predict. This links to the idea that utilization tightens the "coupling" between interconnected networks, and that the dynamics of networks are more attuned to one another at higher utilization rates. We could imagine a similar scenario where resource carriers were fully utilized, so that a sudden export fluctuation in oil could cause ripple effects through the consumer and resource layers. Such ripple might be lessened if there was "slack" capacity that weakened the tension of the ties linking the two networks.

Table 2: Maximum negative impact on consumer-to-consumer links based on r-c network structure in a 3-resource by 3-consumer layer

	(1) max impact of node removal	(2) r-c link dens. > 60%	(3) r-c link dens. > 80%
Shannon index for r-c outdegree	0.542***	1.498***	2.240***
	(106.83)	(151.45)	(99.83)
Shannon index for r-c indegree	-0.205***	-0.583***	-3.232***
	(-33.38)	(-28.93)	(-45.62)
r-c link density	0.375***	0.418***	0.510***
	(432.61)	(312.16)	(185.60)
Constant	-0.578***	-0.540***	1.288***
	(-90.90)	(-24.50)	(16.72)
Observations	809664	148816	28864
R^2	0.328	0.578	0.702

t statistics in parentheses

* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

5 Future Work

Though these first steps are small and measured, this paper hopefully demonstrates the promise of understanding the dynamics of interconnected networks in a simulated setting. Using this approach, we have fine tuned control over the initial conditions and structures that affect our outcomes of interest. Future studies need to look at:

Larger networks: Due to the small number of vertices of the networks utilized in this paper, the opportunity to look at large cascades of failures was prevented. Fu-

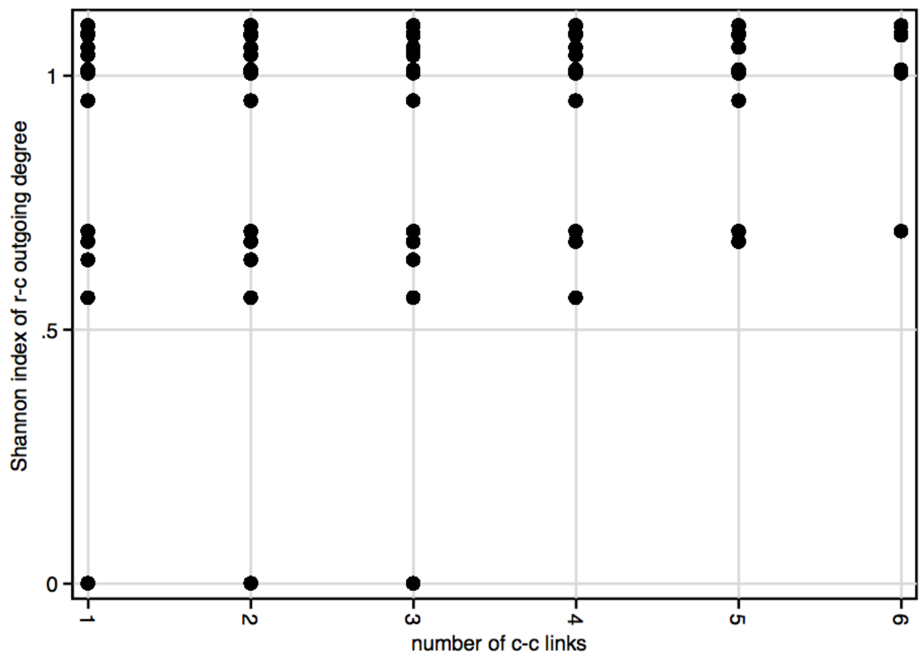

Figure 3: Scatter plot of number of consumer to conusmer links vs Shannon entropy of outgoing degree distribution in the resource-consumer layer for 3x3 graphs

ture studies will look at how small fluctuations resonate through larger interconnected networks.

Real(er) data: Future studies will draw their priors from distributions found in the real world of oil and flight networks, though they could just as easily draw priors from the biological world of predator and prey. Still there is a place for stylized data to get a cleaner sense of the processes at play in dynamical systems.

Tie weights: This study looked at the effects of node removal. However, ties themselves can be lessened in terms of weight, and impacts analyzed. In addition, ties in this study were forced to be of weight one. This stricture can be removed in future studies to more closely resemble resource-consumer networks in nature

Consumer nodes: This study only addressed removal of resource nodes. Future work will address the results of removal of consumer nodes.

Unequal allocation of resources: All nodes were assigned the same production value to allow for a more stable initial environment. Future studies will allocate initial resources from random draws from uniform, normal, and power law distributions. The universe of possible interconnected models will then grow from the initial randomized allocation of resources. This way, we can look at how small changes in initial conditions can have broad impacts.

Tie generation: The current study generates ties in a brute force method that looks at all possible networks. Future studies can generate models using preferential attachment and other means to grow networks.

Adding nodes, ties: Future studies can examine the impacts of adding nodes and ties to existing interconnected networks. This way, we can study questions like the impact of growth in aviation demand in the Global South on the networks of global oil and aviation, or the impact of this change on global greenhouse emissions.

Resource types: This study focused on resources that can be consumed only once. However resource like information exists that can be reused. Also, resources like oil generate flights at first exposure. Other resources require multiple exposures to generate link making activity (such as fads, diseases, etc). Future studies can address these very different dynamics.

Interconnection types: This study focused on a type of interconnection where the consumption of a freely traded resource enables consumers to connect. There could be other interconnection types that bring disparate networks together. For example. financial firms might be linked by transactions that rely on a network of regulators that act as "gatekeepers". Such an interconnection would have very different dynamics than the resource-consumer model.

6 Conclusion

This paper presents a set of tools for studying multiple interconnected networks. Utilizing simulation, a controlled environment can be created in which to understand multinetwork dynamics at small and large scales. The paper uses simulation to look at interconnected networks of resources and consumers. The paper examines when the removal of any single node in the resource network has the largest impact on the consumer network. The paper finds that it's import to focus on the network layer that connects the the separate networks of resources and consumers. In a controlled setting, the paper finds that increasing network density of the resource-consumer layer has a positive impact on minimizing negative consumer impact. Increasing Shannon's diversity index on outgoing degree in the resource-consumer layer minimizes worst case negative impact. Increasing Shannon's diversity on indegree maximizes negative impact. The paper also finds that separate networks are more "attuned" to one another under conditions of higher utilization. There is a wide avenue of future opportunities for further research in this new field of burgeoning interest in the social networks literature.

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