Critical connectivity in banking networks*

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Abstract

The financial crisis of 2007-2009 demonstrated the need to understand the macro-dynamics of interconnected financial systems. A fruitful approach to this problem regards financial infrastructures as weighted directed networks, with banks as nodes and loans as links. Using a simple banking model in which banks are linked through interbank lending, with an exogenous shock applied to a single bank, we find a closed-form analytical solution for the degree at which failures begin to propagate in the network. This critical degree is expressed as a function of four financial parameters: banking leverage; interbank exposure; return on the investment opportunity; and interbank lending rate. While the transition to failure propagation is sharpest with regular networks, we observe it numerically for random and scale-free networks as well. We find that, if the expected number of failures is not strongly dependent on the network topology and is well captured by the notion of critical degree, the frequency of catastrophic cascades (with a single shock inducing all or most banks in the network to fail) tends to be much larger on scale-free networks than on classical random networks. We interpret this finding as a manifestation of the "robust-yet-fragile" property of scale-free networks.

1 Introduction

1.1 Motivation and summary of results

The global turmoil precipitated by the collapse of Lehman Brothers in September 2008 demonstrated the need for a solid understanding of the dynamics of interconnected financial systems and their potential to generate systemic crises (May, 2008; Haldane, 2009; Helbing, 2013). Before 2000, banking models explored motivations and behaviors of individual banks (Diamond and Dybvig, 1983; Santomero, 1984) rather than interbank

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relations (Allen, Babus, and Carletti, 2009). But studying banks in isolation misses the systemic consequences emerging from non-linear interactions of the banks. And even in network research, most theoretical and empirical methods are not suited to predicting failure cascades in economic networks (Schweitzer, 2009).

It has become increasingly evident in the financial sector that shock propagation mechanisms are the very core of the systemic risk concept (DeBandt and Hartmann, 2000). Since the last financial crisis, contagion has come to be seen as perhaps the most important type of systemic failure (Allen, Babus, and Carletti, 2009). However, network models traditionally have not been among the economic models used to understand systemic risk, even though contagion can be created by complex network exposure (DeBandt and Hartman, 2000); only recently have network measures attracted increased attention among researchers and central banks. Some are looking at network techniques to design better banking sector regulations, because the network approach allows identification of conflicts between banks' individual incentives and systemic implications of actions generated by those incentives. Network measures explore more specifically how systemic events unfold (Bisias *et al.*, 2012).

With this work, we try to shed light on the effects of network structure on contagion. We analyze a directed network of interbank lending, with banks as nodes and interbank loans as links. We develop a simple model to obtain a closed-form solution for the expression of the critical degree at which contagion propagates through a network. We also perform simulations to allow for heterogeneity of network structure. The main finding of this study is that the critical degree—which depends on the leverage ratio, lending fraction and interest rate, and can be estimated without detailed knowledge of the network structure—correctly captures the expected number of failures in general networks. The frequency of large deviations (such as catastrophic cascades taking out a large fraction of the network), on the other hand, strongly depends on the network topology

1.2 Literature Review

A new strand of literature is exploring the effects of financial network structures on systemic risk. While some scholars and regulators view interbank lending primarily as an efficient way to cope with liquidity shocks (Freixas and Santomero, 2003), others recognize that interbank lending potentially provides channels for contagion. Allen and Gale (2000), using a simple network model of four banks, show that contagion depends on the structure of banks' interconnections. When there is aggregate shortage of liquidity, complete structures are less prone to contagion than incomplete structures, because cross-holdings better redistribute liquidity. However, Castiglionesi and Navarro (2007), using the same simple four-bank model, showed that greater connectivity could increase contagion risk because banks may make imprudent investments given the greater insurance provided by the financial links. Other studies affirm that complete networks are the most destabilizing because dense linkages facilitate the contagion of shocks (Vivier-Lirimont, 2006; Blume *et al.*, 2011; Battiston *et al.*, 2012; Billio *et al.* 2012). Ladley (2013) concluded that no inter-bank market structure maximizes stability under all conditions. Using a two-period model with two symmetric network structures, Acemoglu *et al.* (2013) find that when the shock is small enough, a complete network structure is more stable than an incomplete one. But when a shock is large, completeness does not guarantee stability and phase transition may occur.

A common network topology in banking is the one with a small number of highly connected financial hubs which seems to create a "robust-yet-fragile" structure, susceptible to rapid transmission of shocks (Haldane and May, 2011): robust because they may withstand many external shocks; fragile because they may suddenly exhibit a cascade of failures. This robust-yet-fragile property is seen in many other complex (scale-free) networks (Watts, 2002). Similarly, in finance, interbank networks act as mutual insurance but, beyond a certain range of connectivity, links amplify shocks (Haldane 2009).

Other studies introduce more realistic balance sheets to study how changes in certain components of balance sheets impact systemic stability. Gai and Kapadia (2010) explore the effects of changes in network structure and asset market liquidity on the probability of contagion derived from counterparty risk. They find that indistinguishable shocks to the network can have vastly different consequences for contagion and identify two phase transitions between which the probability of contagion peaks. In a study of the trade-off between individual and systemic decisions, Beale *et al.* (2011) show that increasing asset diversity across banks makes the system more stable, because it prevents herding behavior that maximizes the probability of systemic collapse.

Several studies analyze the trade-offs between risk-sharing gains in interconnected systems and the costs of increased risk exposure. Allen, Babus, and Carletti (2012) build a two-period model where the links of the banks are the exchange of their asset portfolio, and find that a more clustered network structure is more prone to contagion. Cabrales, Gottardi and Vega-Redondo (2013) analyze the capacity of the different network structures to absorb shocks in a model where the links come from interbank participation on other banks' investment. They find that when shocks follows a fat-tail distribution, extreme segmentation is optimal because it minimizes contagion; while high density is optimal for thin tail distributions, because it achieves the highest risk-sharing. Arinaminathy, Kapadia, and May (2012) add confidence shocks to their model and conclude that the impacts of large, well-connected banks scale more than proportionately with their size. The systemic impact not only depends on the connectivity but also on the level of confidence. Glasserman and Young (2013) introduce bankruptcy costs into a system with confidence loss, and find that the former increases the probability of contagion, while the latter increases its costs. Finally, Elliott, Golub and Jackson (2013) also introduce failure costs and distinguish between first failure, contagion, and propagation. They show that the middle region of connection density is the most vulnerable to contagion. They also find that cascades introduce a moral hazard problem. Firms have an incentive to bail out a large failed bank in order to avoid failure costs to themselves, which then incentivizes failing firms to increase these costs in order to be bailed out.

Most of the literature above uses simulations to study contagion in interbank networks, for two reasons. First, finding analytical solutions on asymmetric network structures with heterogeneous bank sizes, balance sheets, etc. is mathematically hard. Second, data available for empirical studies are limited: nodes may not be easily identifiable, or links may be limited to a particular day, or data sets may omit global exposures, etc.

Despite the data challenges, some studies use empirical data, which are useful for establishing parameters in simulations. Soramaki *et al.* (2007) look at the US interbank payment system and find that the network has both a low average path length and low connectivity; the degree distribution is scale free over a substantial range; the clustering coefficient of the network is 90 times greater than the clustering coefficient of a comparable random network; and the distribution of link weights follows a power law when weighted by the volume of payments and a lognormal when weighted by the value of payments. Degryse and Nguyen (2007) show that Belgium has shifted away from complete networks, which has reduced the contagion risk in this country. Mistulli (2011), with data from the Italian banking system, find that only a small fraction of banks can trigger contagion; which is the classical too-big-to-fail argument. Motivated by the too-big-to-fail debate, Battiston *et al.* (2012) look at the debt exposure among institutions and estimate the systemic importance of a bank using centrality measures. They suggest that the discussion be broadened to institutions too-central-to-fail to account for the systemic importance of highly connected nodes. Arinaminpathy *et al.* (2012) shows that the impact of large well-connected banks on the systemic stability scales more than proportionately with their size, which is similar to the findings in biology, where the role played by superspreaders of infectious diseases is equivalent to the one of large banks in a banking network.

Other studies highlight similarities between networks in finance with networks in ecology and epidemiology. In 1972, May showed that species exhibit a sharp transition from overall stability to instability as the number and strength of interactions increase. Beale *et al.* (2011) find that the tension found in the banking sector between the ideal distribution of assets from an individual stability perspective and from systemic stability perspective is similar to ecological systems, where natural selection leads them to adapt in a similar way while protection of the whole system leads them to diversify. While research on the trade-off between banks' individually optimal behavior and systemic optimal behavior is inconclusive, most studies show that it depends on the structure of the network. We take an additional step towards understanding this trade-off and its implication for financial systemic risk by showing that not only network parameters but also financial parameters such as interbank lending rate, opportunity return rate and leverage are important in assessing contagion risk, with financial parameters being the more important.

1.3 Organization of the paper

The rest of the paper is structured as follows. Section 2 describes the model used in this paper. Section 3 presents the theoretical analysis of the critical degree. Section 4 shows the results of the simulations; and the paper ends with a discussion of the main findings in Section 5.

2 A model of interbank lending

We use a model of interbank payment flows similar to one introduced in (Eisenberg and Noe, 2001), variants of which are commonly used in the recent literature on banking networks, see e.g. (Gai and Kapadia, 2010) and (Acemoglu *et al.*, 2013). Our formulation, however, incorporates two new parameters: accounting leverage Λ and interbank exposure f. As we will see, these allow us to come to grips with the effect of diversification, and in particular of the *mean degree* of the network, on failures propagation.

We analyze a system of N risk-neutral financial institutions ("banks") labelled $i \in \{1 \cdots, N\}$ operating in a simple two-period economy. Each bank has generic assets and liabilities in addition to interbank loans. At $t = 0$, each bank has an investment opportunity and uses its liabilities to fund the investment. At $t = 1$, the investment project yields a return of R; and the interbank debts are repaid with a return of r, where $R > r > 1$.

The senior liabilities do not bear any interest and must also be repayed at $t = 1$.

In our model, one bank, bank i, receives an exogenous shock on the return on investment such that $R_i = 0$. If the total revenues of the shocked bank are less than the debt to be repaid, the bank defaults. This individual default may create distress on its creditors since the failed bank may be unable to repay its debt in full. Consequently, other banks may be affected and a cascade of failures could occur. In this way an individual bank failure may become a systemic banking failure. We study what factors determine this systemic failure.

2.1 Balance sheets and financial ratios

The banking network consists of interbank loans, represented as l_{ij}, with $l_{ij} > 0$ when there is a directed link from bank i to bank j in the network and $l_{ij} = 0$ otherwise.[1]

Each bank i is initialized with an asset portfolio consisting of these interbank loans, $l_i = \sum_{j \neq i} l_{ij}$, as well as other liquid assets (cash, bonds, etc.) λ_i and illiquid assets (buildings, etc.) ι_i:

$$\text{asset}_i = l_i + \lambda_i + \iota_i. \tag{1}$$

Total Assets	Total Liabilities
liquid assets λ_i	senior liabilities σ_i
illiquid assets ι_i	interbank borrowings b_i
interbank loans l_i	capital K_i

Figure 1: Balance sheet of bank i.

We assume that the fractions of each type of asset are constant throughout the network. These fractions are defined as

$$\begin{aligned} f &= l_i/\text{asset}_i \\ f^{(\lambda)} &= \lambda_i/\text{asset}_i \\ f^{(\iota)} &= \iota_i/\text{asset}_i. \end{aligned}$$

The liabilities of each bank i, in turn, consist of interbank borrowings $b_i = \sum_{j \neq i} l_{ji}$ and senior liabilities σ_i:

$$\text{liab}_i = b_i + \sigma_i. \tag{2}$$

The senior liabilities consist of debts which take priority for repayment, such as deposits. We assume that the leverage Λ is fixed, and is defined according to the capital K_i as

$$\Lambda \equiv \text{assets}_i/K_i \tag{3}$$

$$K_i \equiv \text{asset}_i - \text{liab}_i, \tag{4}$$

implying that the senior liabilities σ_i can be expressed in terms of constants and each bank's lending and borrowing l_i and b_i as

$$\sigma_i = \frac{\Lambda - 1}{\Lambda f} l_i - b_i. \tag{5}$$

[1] A notational note: hereafter the first (resp. second) index always denotes the source (resp. target) of a money flow.

2.2 Investment opportunities

Next, each bank uses the entirety of its liabilities to invest in an external market, obtaining in return on its investment[2]

$$\rho_i = (R_i - 1)\,\mathrm{liab}_i = (R_i - 1)\,\frac{\Lambda - 1}{\Lambda}\,\mathrm{asset}_i. \tag{6}$$

Here R_i is an interest rate, which may be larger than one for a succesful investment or smaller than one for an unsuccessful investment, or "shock".

2.3 Repayment equation

Third, after the investments are made and their returns collected, all banks repay first their senior liabilities σ_i then their junior (interbank) liabilities with an interest rate r. Repayments are made using their return on investments ρ_i, liquid assets λ_i and incoming repayments x_{ji}, as follows:

- *Full repayment*: if $\rho_i + \lambda_i - \sigma_i + \sum_{j\neq j} x_{ji} \geq rb_i$, bank i repays its junior debt in full, hence for each $j \neq i$
$$x_{ij} = rl_{ji},$$
- *Partial default*: if $0 < \rho_i + \lambda_i - \sigma_i + \sum_{j\neq j} x_{ji} < rb_i$, bank i repays only a fraction of its junior liabilities on a *pro rata* basis, hence for each $j \neq i$
$$x_{ij} = \frac{l_{ji}}{b_i}\left(\rho_i + \lambda_i - \sigma_i + \sum_{j\neq i} x_{ji}\right)$$
- *Complete default*: if $\rho_i + \lambda_i - \sigma_i + \sum_{j\neq j} x_{ji} \leq 0$, bank i repays nothing at all, hence $x_{ij} = 0$ for each $j \neq i$.

These rules are summarized by the equation

$$x_{ij} = \frac{l_{ji}}{b_i}\max\left\{\min\left\{\rho_i + \lambda_i - \sigma_i + \sum_{j\neq i} x_{ji}, rb_i\right\}, 0\right\}. \tag{7}$$

Defining $x_i \equiv \sum_{j\neq i} x_{ij}$, we say that bank i is in *partial default* when $0 < x_i < rb_i$ and in *complete default* when $x_i = 0$; when a bank i can just repay its interbank borrowings, i.e. when

$$\rho_i + \lambda_i - \sigma_i + \sum_j \frac{l_{ij}}{b_j}\,x_j = rb_i, \tag{8}$$

we say that i is *critical.*

After all repayments are made, bank i has a new capital

$$K_i' = \rho_i + \lambda_i + \iota_i - \sigma_i + \sum_{j\neq i}(x_{ji} - x_{ij}). \tag{9}$$

We call "safe" the banks i such that $K_i' \geq 0$, and "failed" the ones such that $K_i' < 0$. The remainder of this manuscript is dedicated to the study of the number F of failed banks as a function of the financial parameters (interbank exposure f, accounting leverage Λ, interest rates R and r) and of the network topology. For simplicity we will neglect the effect of illiquid assets, i.e. we will take $\iota_i = 0$ so that $f^{(\lambda)} = 1 - f$.

[2] If $\sigma_i < 0$, we take $\rho_i = (R_i - 1)b_i$; equivalently, the return is defined by $\rho_i = (R_i - 1)\max\{\mathrm{liab}_i, b_i\}$.

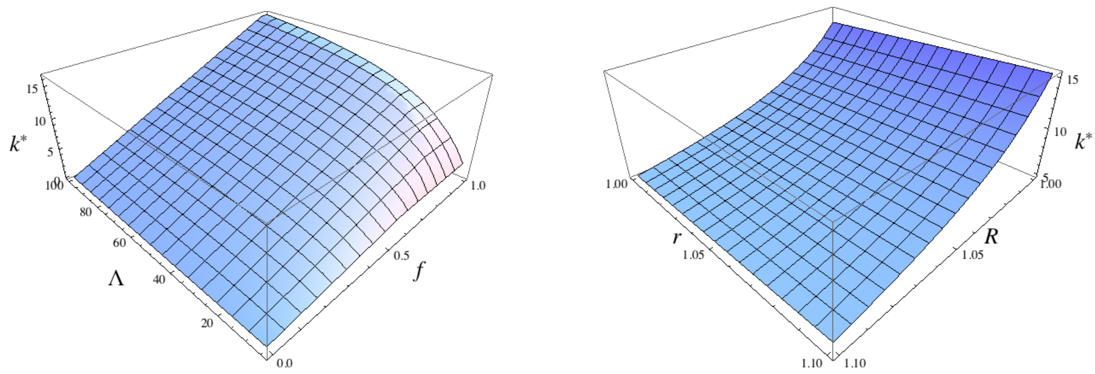


Figure 2: The critical degree (15) as a function of the interbank exposure f and of the leverage factor Λ, for $R = 1.05$, $r = 1.01$ (left) and as function of R and r for $f = 70\%$ and $\Lambda = 20$ (right).

3 Critical diversification and expected number of failures

In this section we study the relationship between failures contagion and degree in the network (aka diversification). Specifically, we provide an explicit formula (in terms of the financial ratios f and Λ and of the interest rates R and r) for the *critical degree* k^*, such that failures extend to non-shocked banks only if $k \leq k^*$. To this aim, let us begin by considering the special case of regular networks.

3.1 Regular networks

A *regular* network is a network where the in-degrees $\vec{k}_i$ and out-degrees $\overleftarrow{k}_i$ of all banks i are equal to the same value k. It follows from this assumption that $l_i = b_i = k$, hence the repayment equation

$$x_i = \left[\min \left\{ \rho_i + \lambda_i - \sigma_i + \sum_{j \leftarrow i} \frac{x_j}{b_j}, r b_i \right\} \right]^+ \tag{10}$$

becomes

$$x_i = \left[\min \left\{ \eta_i k + \sum_{j \leftarrow i} \frac{x_j}{k}, r k \right\} \right]^+ \tag{11}$$

where $\eta_i = (2 - \Lambda)/\Lambda f$ if i is the shocked bank and $\eta_i = [(R-1)\Lambda + (2-R)]/\Lambda f$ else.

To proceed, we now make the further assumption that *all banks at a given distance d from the shocked bank repay the same amount x_d*. This allows us to write

- for the shocked bank:
$$x_0 = \left[\frac{2 - \Lambda}{\Lambda f} k + x_1 \right]^+, \tag{12}$$

- for the first neighbors of the shocked bank:
$$x_1 = \left[\min \left\{ \frac{(R-1)\Lambda + (2-R)}{\Lambda f} k + \frac{x_0 + c_0(k-1)x_1 + (1-c_0)(k-1)x_2}{k}, r k \right\} \right]^+, \tag{13}$$

  where c_0 is the local clustering coefficient of the shocked bank.

The critical degree k^* is defined by the condition that only the shocked bank defaults, and all its first neighbors are critical. This means that $x_1 = x_2 = rk^*$ and

$$\eta k^* + \frac{x_0 + c_0(k^* - 1)rk^* + (1 - c_0)(k^* - 1)rk^*}{k^*} = rk^*. \tag{14}$$

Equations (12) and (14) can be solved for k^*, giving

$$k^* = \left(\frac{r - [r - (\Lambda - 2)/(\Lambda f)]^+}{(R - 1)\Lambda + (2 - R)} \right) \Lambda f. \tag{15}$$

We can distinguish two cases.

- If the shocked bank is in complete default ($f < (\Lambda - 2)/\Lambda r$), the critical degree depends on the interbank interest rate r, reading

$$k^* = \frac{rf\Lambda}{(R - 1)\Lambda + (2 - R)}.$$

- If the shocked bank is only in partial default ($f \geq (\Lambda - 2)/\Lambda r$), the critical degree depends neither on the interbank interest rate r nor on the loan fraction f, reading

$$k^* = \frac{\Lambda - 2}{(R - 1)\Lambda + (2 - R)}.$$

Observe that, in the limit of large leverage ($\Lambda \gg 1$) and large interbank exposure ($f \to 1$), the expression (15) reduces to

$$k^* \simeq \frac{1}{R - 1}. \tag{16}$$

Thus, in this limit, the critical degree is just the inverse return on investment, and can therefore take large values, of the order of one hundred. The general dependence of k^* on the interbank exposure f and the leverage factor Λ is plotted in Fig. 2.

In a regular network, we therefore predict that $F = 1$ for $k \geq k^*$ (only the shocked bank fails), $F = 1 + k$ for $k < k^*$ but k not too small (all first neighbors fail, and no other bank fails), and $F > 1 + k$ for k very small (all first neighbors and some higher neighbors fail). This is confirmed in Fig. 3, where the number of failures in regular networks with $N = 20$ was computed numerically.

3.2 More general networks

The case of regular networks forms the basis of a mean-field-type approximation of the expected number of failures $\langle F \rangle$ for more general random networks, including scale-free ones. This approximation is based on the following assumptions:

- In any directed random network, the average in-degree $\langle \vec{k}_i \rangle$ and the average out-degree $\langle \overleftarrow{k}_i \rangle$ are equal to the mean degree k. Here, we shall assume that $\vec{k}_i = \overleftarrow{k}_i \equiv k_i$ for each bank i, even though k_i may depend on i.
- On average, each bank i with degree k_i reacts to the shock of another bank as if *all* banks had degree k_i; in particular, the first neighbors of the shocked bank will fail if $k_i < k^*$, and else they will remain safe.

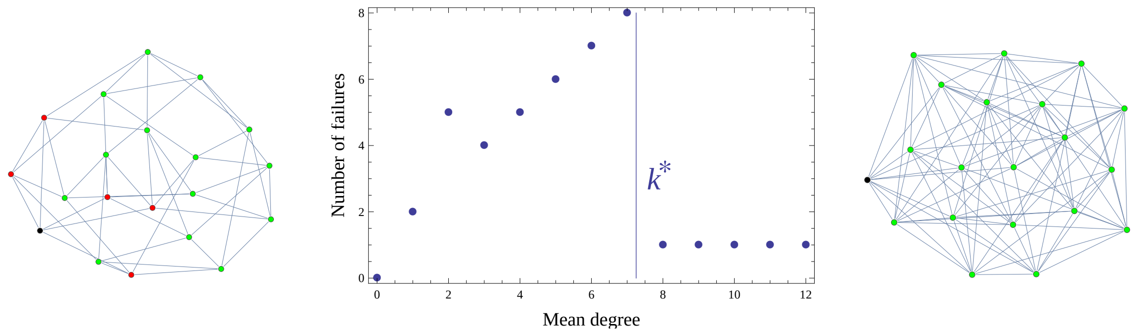


Figure 3: Failures in a regular network. Center: number of failures in a regular network with $N = 20$ banks (with $R = 1.05$, $r = 1.01$, $f = .7$, $\Lambda = 20$) as a function of the degree k; the vertical line corresponds to the critical degree k^* obtained in (15). Note the second maximum at $k = 2$, corresponding to the extension of failures to second neighbors of the shocked bank. Sides: sample regular networks with $k = 5$ (left) and $k = 10$ (right); the shocked bank is represented in black, the failing banks in red and the safe banks in green.

Denoting $f(k)$ the cumulative degree distribution (the probability that a given bank has an in-degree and out-degree $\leq k$), z the mean degree and z_2 the mean number of second neighbors, we are thus led to the expression

$$\langle F \rangle \simeq 1 + kf(k^*) \tag{17}$$

where k is the mean degree of the network.[3] We test the validity of this—admittedly rough—approximation in the next section.

4 Comparison of different network topologies

4.1 Models of random graphs

In order to assess the impact of the network topology on the number of failures, we generated various directed and undirected random networks and ran numerical simulations of the repayment equation.

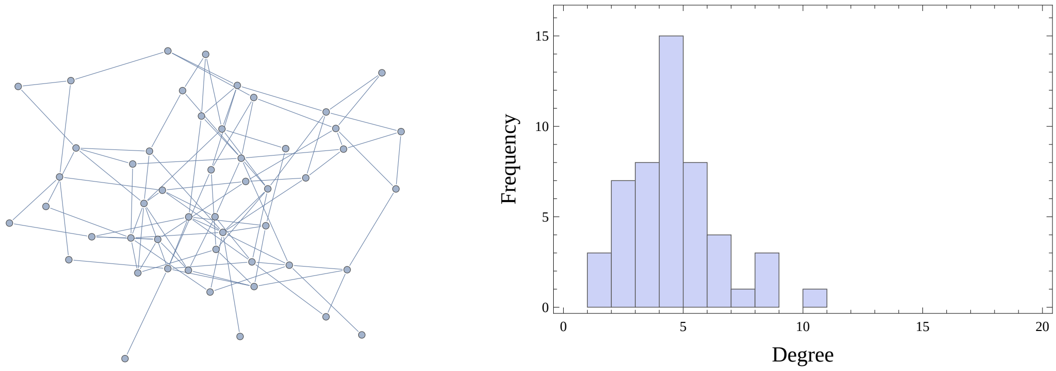
- *Undirected networks.*

  Undirected networks correspond to the case where, whenever bank i lends money to bank j, bank j also lends money to bank i. Although unrealistic, this situation is the simplest one to investigate, both analytically and numerically. Furthermore, it allows us to use familiar models of random networks, such as the *Erdös-Rényi* model (Erdös and Rényi, 1959) and the *Barabási-Albert* model (Barabási and Albert, 1999).

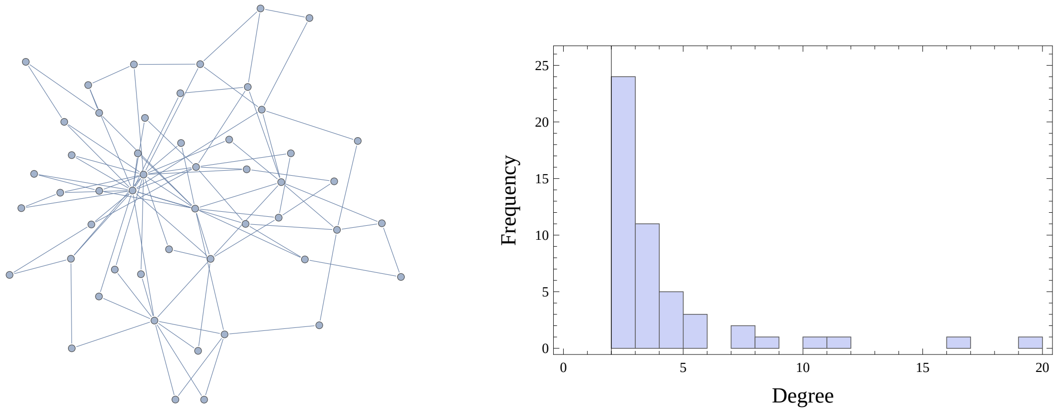
  - **Erdös-Rényi networks**. Starting from N initially unconnected nodes, each pair of nodes is connected by an undirected link with probability p. The resulting network has a Poissonian degree distribution with mean $k = p(N-1)$. See Fig. 4a.

[3] This formula could in principle be refined as $\langle F \rangle \simeq 1 + k_2 f(k_2^*) + \cdots$, where k_2 is the mean number of second neighbors and k_2^* the "second critical degree", corresponding to the defined as the maximum degree such that failures extend beyond the first neighbors of the shocked bank (see the second maximum in Fig. 3).

– **Barabási-Albert networks**. Given m_0 initial nodes, we attach at each time step a new node to $m \leq m_0$ existing nodes, according to the rule of *preferential attachment*, meaning that a new node is attached to an old one i with probability $k_i / \sum_j k_j$. For sufficiently large N, this yields a power-law degree distribution with exponent -3. See Fig. 4b.



(a) Erdös-Rényi network



(b) Barabási-Albert network

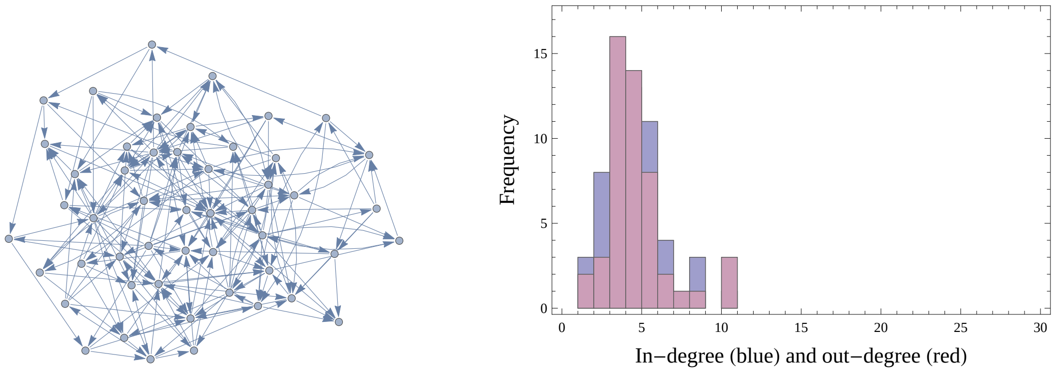
Figure 4: Sample undirected networks with $N = 50$ and $k = 4$ (left) and the histograms of their vertex degrees (right).

- *Directed networks.*

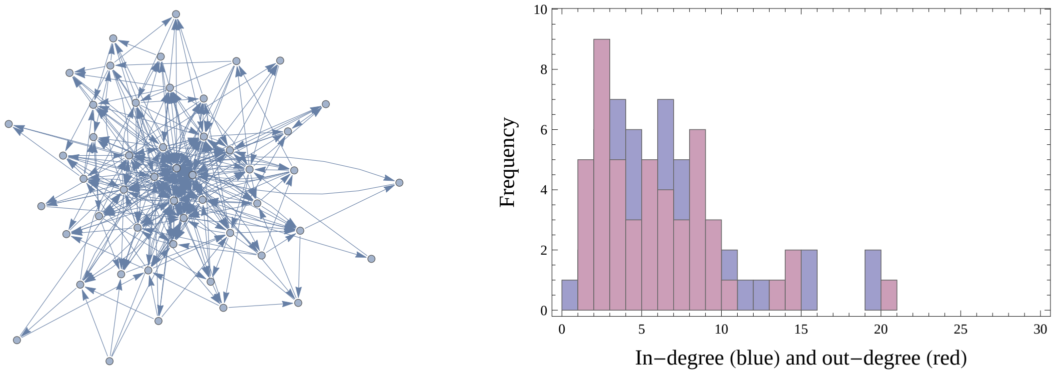
  Directed networks correspond to the general case where bank i may lend to bank j even though bank j does not lend to bank i. While it is easy to modify the Erdös-Rényi model to incorporate directedness, there is no obvious directed generalization of the Barabási-Albert model. Scale-free directed networks, however, can easily be generated using the so-called "static" model (Goh *et al.*, 2001).

  – **Erdös-Rényi directed networks**. Starting from N initially unconnected nodes, each ordered pair of nodes is connected by an directed link with probability p. The resulting network has Poissonian in-degree and out-degree distributions with mean $k = p(N-1)$. See Fig. 5a.

  – **Goh-Kahng-Kim "static" directed networks**. Starting from N initial nodes i, define two probability distributions $p_{\text{in}}(i) = i^{-\alpha_{\text{in}}} / \sum_{j=1}^{N} j^{-\alpha_{\text{in}}}$ and $p_{\text{out}}(i) = i^{-\alpha_{\text{out}}} / \sum_{j=1}^{N} j^{-\alpha_{\text{out}}}$, where $0 \leq \alpha_{\text{in}}, \alpha_{\text{out}} < 1$. Then draw a node i at random from the distribution p_{in} and a node i' at random from the distribution p_{out}; if $i \neq i'$, assign a directed link from i to i'; else do nothing. Repeat this operation Nk times, dropping multiple edges when they occur. The resulting network has power-law in-degree and out-degree distributions, with respective exponents $(\alpha_{\text{in}} + 1)/\alpha_{\text{in}}$ and $(\alpha_{\text{out}} + 1)/\alpha_{\text{out}}$. See Fig. 5b.



(a) Directed Erdös-Rényi network



(b) Directed static scale-free (with $\alpha_{\text{in}} = \alpha_{\text{out}} = .9$) network

Figure 5: Sample undirected networks with $N = 50$ and $k = 4$ (left) and the histograms of their vertex degrees (right).

4.2 Distribution of failures: numerical results

For each (undirected and directed) random networks models, we simulated the model of interbank payment flows of sec. 2 using Mathematica and Matlab, and plotted the number of failures F as a function of the mean degree k for various values of the financial parameters R, r, f and Λ. We also developed a graphical tool to visualize the propagation of failures across the network, which proved very useful in guiding our intuition (see Fig. 6).

Our results, presented in Fig. 7 - 9, can be summarized as follows:

- *Existence of a transition.* In all cases, we observed a (smooth) transition between a no-contagion regime for $k \gg k^*$ to a contagion regime for $k \precsim k^*$. At low degree ($k \simeq 0$), the isolation of the shocked bank ensures that failures do not propagate (in the Erös-Rényi case, $k = 1$ is the percolation threshold).
- *Validity of the mean-field approximation.* In spite of its simplicity, we found that our mean-field approximation is in qualitative agreement with the numerical results for

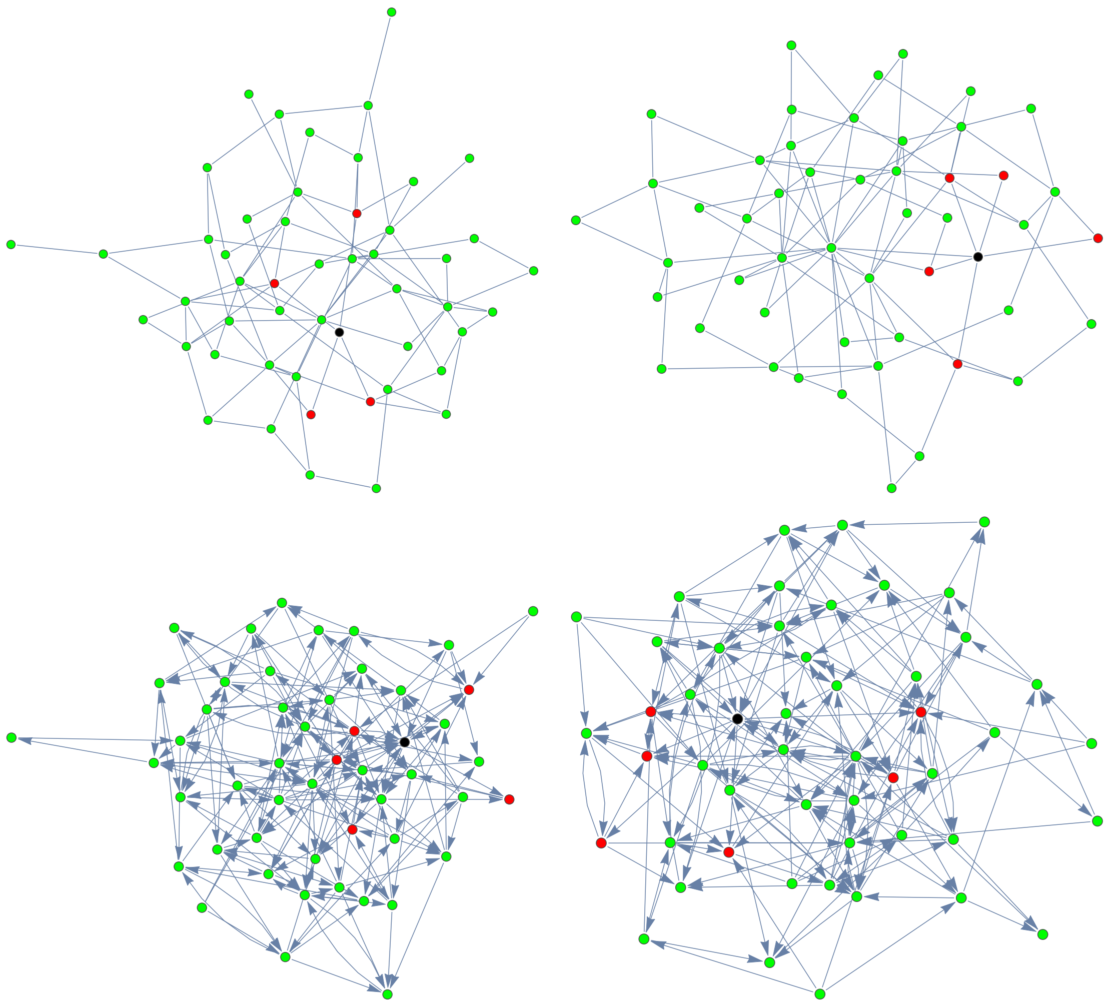


Figure 6: Visualization of failure propagation in various types of networks of $N = 50$ with mean (in-)degree $k = 4$ (from top left to bottom right: Erdös-Rényi, Barabási-Albert, directed Erdös-Rényi, directed static scale free). The black node indicates the shocked bank, the red nodes the failed banks, the green nodes the safe banks. Here $R = 1.05$, $r = 1.01$, $f = 70\%$ and $\Lambda = 20$.

the mean number of failures, see. Fig. 7 and Fig. 8. In the Barabási-Albert case, formula (17) slightly overestimates the mean number of failures in the contagion regime; in the directed static scale-free case, on the contrary, (17) is an underestimation.

- *Large deviations and catastrophic failures.* The mean number of failures by itself gives us little information about the probability of catastrophic cascades—failures taking out a significant fraction of the network. We found that, while the mean number of failures is slightly smaller in the scale-free (Barabási-Albert and directed static) cases, the probability of catastrophic cascades is much larger than in the Poissonian case (Erdös-Rényi models), see Fig. 9. This is an illustration of the "robust-yet-fragile" nature of scale-free networks, which can be traced back to the presence of many low-degree, low-threat banks (robustness) as well as a few high-degree, high-threat banks (fragility).

All in all, our results illustrate that, in spite of their "complexity" and their "robust-yet-fragile" character (observed in Fig. 9), the *expected* behavior of scale-free interbank networks vis-à-vis financial shocks can be understood analytically through a single function

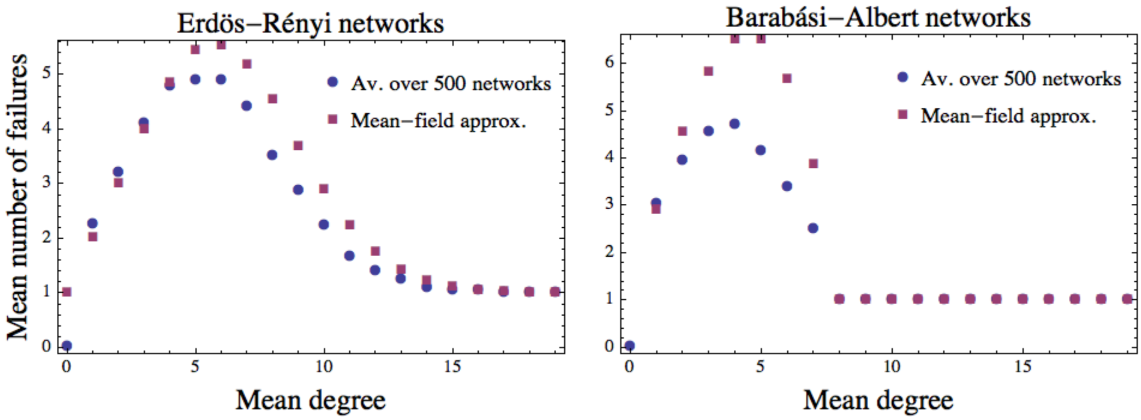


Figure 7: Expected number of failures in undirected Erdös-Rényi (left) and Barabási-Albert (right) networks with $N = 50$ banks, for $R = 1.05$, $r = 1.01$, $f = 70\%$ and $\Lambda = 20$.

of the financial parameters, namely $k^*(f, \Lambda, R, r)$. This finding is the main contribution of this work.

5 Conclusion

This research attempts to understand the effects of network structure and financial parameters on the propagation of contagion in financial networks. We explored this problem using a mean-field type approximation, which allowed us to obtain a closed-form analytical solution for the degree at which failures begin to propagate in the network. This critical degree depends on the financial parameters of the model; that is, leverage Λ; interbank exposure f; return on the investment opportunity R; and interbank lending rate r. Our computer simulations tested the robustness of our results on various types of networks. While failure propagation shows a sharp transition at critical degree for regular networks, the transition is more gradual for general random networks because of the non-uniformity of the degree distribution. We provide a rough but qualitatively correct estimate of the expected number of failures in such more general cases.

It has often been stressed that real-world networks, owing to their scale-free nature, exhibit a robust-yet-fragile character. (In such networks, most shocks affect low-degree

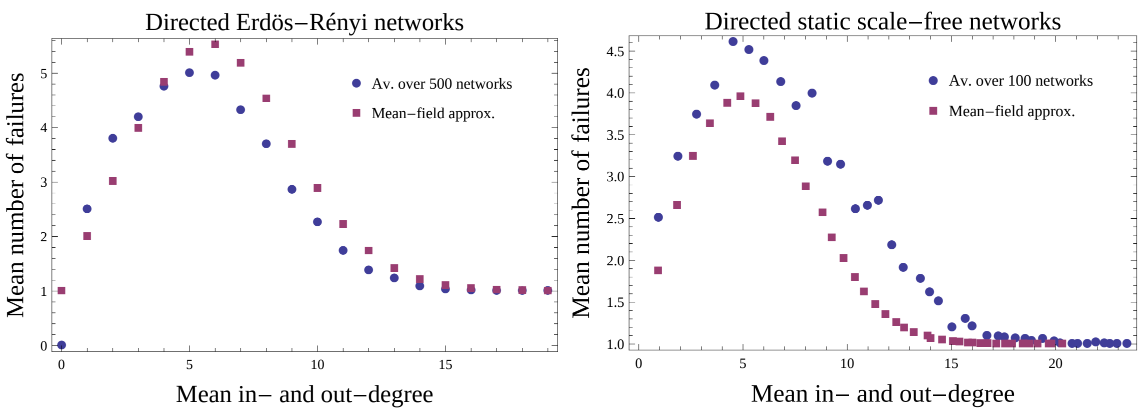


Figure 8: Expected number of failures in directed Erdös-Rényi (left) and directed static scale-free (right) networks with $N = 50$ banks, for $R = 1.05$, $r = 1.01$, $f = 70\%$ and $\Lambda = 20$.

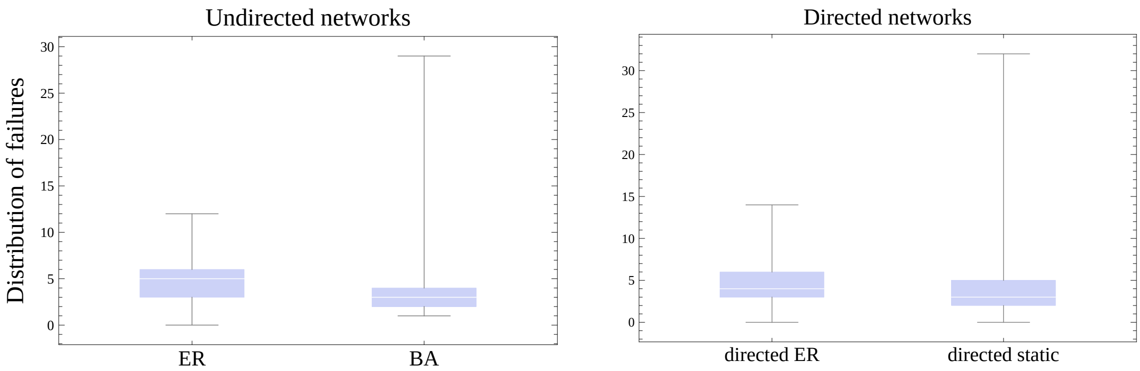


Figure 9: Distribution of failures for undirected Erdös-Rényi and Barabási-Albert networks (left), and directed Erdös-Rényi and directed static scale-free networks (right) of $N = 50$ banks with mean (in- and out-)degree $k = 4$, for $R = 1.05$, $r = 1.01$, $f = 70\%$ and $\Lambda = 20$.

nodes and are effectively absorbed, but shocks to high-degree hubs can lead to catastrophic cascades.) We observed these large fluctuations in our model as well, confirming the strong connection between degree and failure distributions.

Besides refining our mean-field approximation to put this connection between degree and failures distribution on stronger mathematical footings, future work could include various extensions of the model itself. Indeed, while this research considers a simplified situation where an exogenous shock affects a single bank, one may wish to consider consecutive shocks to multiple banks. Further, variability in interest rates may be incorporated in the model by drawing investment rate R from a probability distribution, capturing the heterogeneity in the investments of each bank. Other extensions could include the effects of out-degree on the loan sizes, which appear to take the form of a power law (Soromäki *et al.* 2007). Assets classes and multiple time periods could be introduced to make the study more comprehensive and realistic.

Nonetheless, establishing a critical degree for the spread of financial contagion in a banking network is significant. Determining and regulating interbank loan networks in real time is a challenge, due to confidentiality concerns and rapidly evolving loan portfolios. As critical degree seems to depend on financial variables in a definite way, regulators should pay particular attention to the monitoring and control of these variables to minimize the probability of systemic failures in the banking system.

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